

Semilinear wave equations on the Witten bubble spacetime

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Semiclassical instability of the Kaluza–Klein vacuum

- Kaluza–Klein $\mathbf{R}^{3+1} \times S^1$ is a flat solution to the Einstein vacuum equations that is classically stable
[Huneau–Stingo–Wyatt '25]
- Semiclassical instability [Witten '82]: a uniformly accelerating bubble forms where the compact dimension degenerates

The Witten bubble metric

$$\mathcal{M} = \mathbf{R}_\tau \times \mathbf{R}_{x,\theta}^2 \times S^2$$

$$g = -\cosh^2 x \, d\tau^2 + \cosh^2 x \, dx^2 + \cosh^2 x \cosh^2 \tau g_{S^2} + \tanh^2 x g_{S^1}.$$

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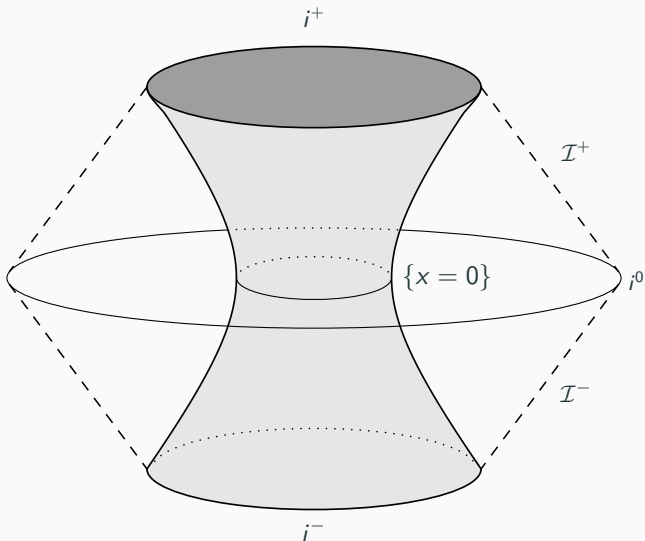
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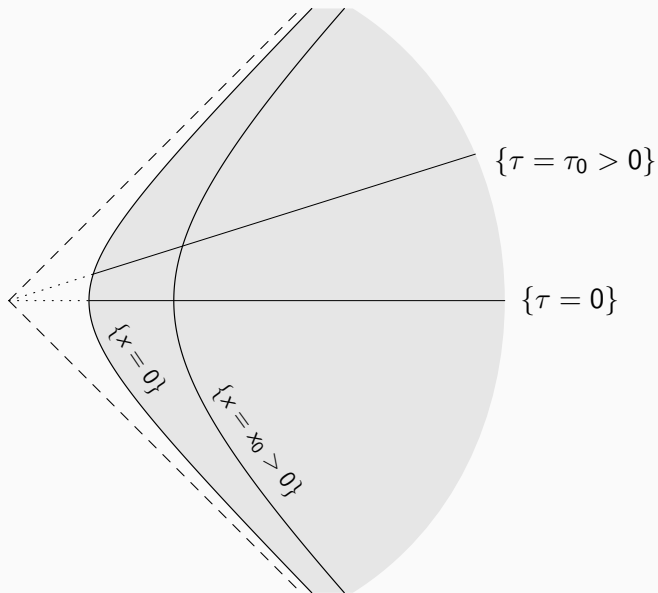
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- Kaluza–Klein as $x \rightarrow \infty$ (at spatial and null infinity)
- $\mathbf{R}^2$ with an expanding S^2 as $x \rightarrow 0$
- $(2+1)$ de Sitter at $x = 0$

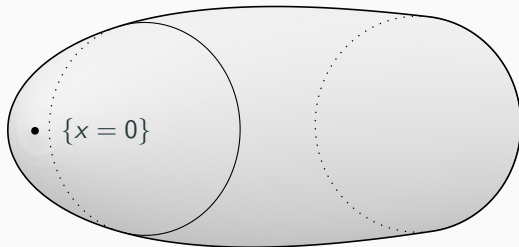
Penrose-type diagram: 3D



Penrose-type diagram: 2D



The asymptotically cylindrical R^2 factor of the spacetime



The asymptotic geometry of the spacetime

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- Kaluza–Klein as $x \rightarrow \infty$ (at spatial and null infinity)
 - $\partial_\tau \sim r\partial_t + t\partial_r$ is timelike and $\partial_x \sim t\partial_t + r\partial_r$ is spacelike
 - $\partial_\tau + \partial_x \sim V\partial_V$ and $\partial_\tau - \partial_x \sim U\partial_U$ for Minkowskian
(U, V) $\sim (-e^{-(\tau-x)}, e^{(\tau+x)})$

Linear theory

- Goal: study $\square_g \phi = 0$
- Work with $\psi := r\phi$
- Split $\psi = \psi_0 + \psi_{\geq 1}$, where ψ_0 is $U(1)$ -symmetric

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Warning

Energy estimates in terms of ϕ cannot hold!

Main theorem on linear waves

	ψ_0	$\psi_{\geq 1}$
Character	massless wave	wave with effective mass
Spatial decay	bounded: $O(1)$	decays: $O((\cosh x)^{-1/2})$
Derivative decay	$\partial_x \psi_0 = O(1)$	$\partial_x \psi_{\geq 1} = O((\cosh x)^{1/2})$
Time decay	decays	may oscillate
Radiation field	may be nonzero	vanishes
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Key observation

Better time decay and better spatial decay occur in different parts of the solution!

The origin of the effective mass

- Fourier expansion in S^1 produces a Klein–Gordon term with mass growing towards spatial infinity
- Finite-energy solutions $\psi_{\geq 1}$ decay quickly in space
- Confining potential for $\psi_{\geq 1}$ admits finite-energy eigenfunctions, so there are periodic solutions

Aside: weak trapping

Generic null geodesics without S^1 -angular momentum escape to $x = \infty$.

Null geodesics with S^1 -angular momentum have bounded x .

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This is not conventional trapping

Everything in the spacetime is going to $r = \infty$!

Linear estimates: the linear operator

Work with ψ and $P\psi := \cosh^2 x \cdot r\Box\phi$.

$$\begin{aligned} P\psi = & -\partial_\tau^2 + \partial_x^2 + \frac{1}{\sinh x \cosh x} \partial_x \psi - \frac{1}{\cosh^2 x} \psi \\ & + \frac{1}{\cosh^2 \tau} \Delta_{S^2} \psi + \frac{\cosh^2 x}{\tanh^2 x} \partial_\theta^2 \psi \end{aligned}$$

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The 2D structure near $x = 0$

Recall $\Delta_{\mathbb{R}^2}$ acting on radial functions is $\partial_x^2 + \frac{1}{x} \partial_x$.

Linear estimates: the ∂_τ -energy

$$E_T[\psi](\tau) := \int_{\Sigma(\tau)} \left[(\partial_\tau \psi)^2 + (\partial_x \psi)^2 + \frac{1}{\cosh^2 x} \psi^2 \right. \\ \left. + \frac{1}{\cosh^2 \tau} |\nabla_{S^2} \psi|^2 + \frac{\cosh^2 x}{\tanh^2 x} (\partial_\theta \psi)^2 \right] \tanh x \, d\mu$$

for $d\mu = dx \, d\omega \, d\theta$.

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for $d\mu = dx \, d\omega \, d\theta$. There is an energy estimate

$$\begin{aligned} E_T[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} \int_{\Sigma(\tau)} \frac{\tanh \tau}{\cosh^2 \tau} |\nabla_{S^2} \psi|^2 \tanh x \, d\mu \, d\tau \\ \lesssim E_T[\psi](\tau_1) + \int_{\tau_1}^{\tau_2} \int_{\Sigma(\tau)} |\partial_\tau \psi| |P\psi| \tanh x \, d\mu \, d\tau \end{aligned}$$

for $0 \leq \tau_1 \leq \tau_2$.

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Remark

Bulk term gives exponential decay in s , which is polynomial decay on Minkowski.

Semilinear wave equations

Main theorem

Theorem (global existence)

Consider $\square_g \phi = F(\nabla \phi)$. Suppose that

- F is quadratic and satisfies a version of the null condition,
- and the non- $U(1)$ -symmetric part of the coefficients of F decay suitably fast in space.

Then small-data solutions exist globally in time. If the data and the coefficients of the nonlinearity are $U(1)$ -symmetric, then the solution has improved time decay (in $u = \tau - x$).

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Example: $F(\nabla \phi) = f \cdot g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$.

Proof strategy

- Rewrite in terms of ψ and $P\psi = \cosh^2 x \cdot r \square \phi$.
- Then $P\psi$ sees $r \partial \phi \cdot \partial \phi \sim r^{-1} [\partial \psi \cdot \partial \psi + \psi \cdot \partial \psi + \psi \cdot \psi]$.

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Plenty of room in time, no room in space.

The worst term

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Key observation

$(\partial(f \cdot \psi \cdot \partial_x \psi))$ only produces $f_{\geq 1} \cdot (\partial \psi)_0 \cdot \partial_x \psi_{\geq 1}$
(not $f_0 \cdot (\partial \psi)_0 \cdot \partial_x \psi_{\geq 1}$), and $f_{\geq 1}$ decays better in space.

Takeaways and outlook

Takeaways

Compact dimension creates strong distinction between zero and non-zero S^1 -modes.

- ψ_0 is better in time and worse in space, and $\psi_{\geq 1}$ is better in space and worse in time
- Origin: weak trapping for null geodesics
- Interplay between spatial and time decay allows global existence for nonlinear wave equations

Stability?

Until next time...

Thank you!