DECAY WITHOUT A RATE

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Notation. Decompose a function $f : \mathbf{R}_t \times S^2 \to \mathbf{R}$ as a sum of L^2-orthogonal spherical modes $f = \sum_L f_L$.

If the spherical modes decay with a uniform rate and are summable uniformly in time and over the sphere, we can deduce decay without a rate for f.

Lemma 1. *Suppose that:*

(i) $\|f_L\|_{L^\infty(S^2)}(t) \leq C_L \epsilon(t)$ *for constants* $C_L > 0$ *and* $\epsilon(t) \to 0$,

(ii) $\|f_L\|_{L^\infty(\mathbf{R}\times S^2)} \in \ell^1_L$.

Then f *decays in time:*

$$\lim_{t\to\infty} \|f_L\|_{L^\infty(S^2)}(t) = 0. \tag{1}$$

Proof. Since $\epsilon(t) \to 0$, we can construct $L(t)$ (e.g. piecewise constant) so that $\epsilon(t) \sum_{L \leq L(t)} C_L \to 0$ and $L(t) \to \infty$. Now

$$\|f\|_{L^\infty(S^2)}(t) \leq \sum_L \|f_L\|_{L^\infty(S^2)}(t) \leq \epsilon(t) \sum_{L\leq L(t)} C_L + \sum_{L \geq L(t)} \|f_L\|_{L^\infty(\mathbf{R}\times S^2)}. \tag{2}$$

By construction of $L(t)$ and assumption (ii), the right side vanishes as $t \to \infty$. □

Even without uniform estimates for f_L, one can still deduce decay without a rate for f, given additional control of derivatives as well as energy boundedness.

Lemma 2. *Write* $E_k[f](t)$ *for* $E_k[f(t,\cdot)] := \|\Omega^{\leq k} f(t,\cdot)\|_{L^2(S^2)}$. *Suppose that:*

(i) *each spherical mode decays together with its first two angular derivatives, i.e.* $\lim_{t\to\infty} \Omega^k f_L(t,\omega) = 0$ *for each* $\omega \in S^2$ *and* $k \leq 2$,

(ii) *energy boundedness holds in the sense that* $E_k[f_L](t) \lesssim E_k[f_L](0)$ *and* $E_k[f](t) \lesssim E_k[f](0)$ *for* $k \leq 4$, *where the implied constants are independent of* t *and* L,

(iii) *the initial data is bounded, in the sense that* $E_4[f](0) < \infty$.

Then f *decays in time:*

$$\lim_{t\to\infty} \|f\|_{L^\infty(S^2)}(t) = 0. \tag{3}$$

Proof. Sobolev embedding on the sphere and L^2-orthogonality of the spherical modes imply

$$\|f\|^2_{L^\infty(S^2)}(t) \lesssim E_2[f](t) = E_2[\textstyle\sum_L f_L](t) = \displaystyle\sum_L E_2[f_L](t),$$

where we used assumptions (ii) and (iii) to infer that $\Omega^{\leq 2} f(t,\cdot) \in L^2(S^2)$ and justify the interchange of integral and limit in the final equality. By the energy boundedness assumption (ii), L^2-orthogonality, and the boundedness of initial data assumption (iii), the summand on the right is dominated uniformly in time by the ℓ^1_L quantity $E_2[f_L](0)$. The dominated convergence theorem therefore allows the interchange of a limit in time and the sum in L:

$$\lim_{t\to\infty} \|f\|^2_{L^\infty(S^2)}(t) \lesssim \sum_L \lim_{t\to\infty} E_2[f_L](t).$$

Finally, assumptions (ii) and (iii) imply that $\Omega^{\leq 2} f(t,\omega)$ is dominated uniformly in time by $E_4[f](0)$ (which is in $L^2(S^2)$), which justifies the interchange of the limit in time and the integral over the sphere implicit in E_2. Using assumption (i) now completes the proof. □

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