ENDPOINT SOBOLEV EMBEDDING

1. Failure of embedding $\dot{W}^{1,n}(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$

The putative endpoint homogeneous Sobolev embedding $\dot{W}^{1,n}(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$ fails when $n \geq 2$.[1] A counterexample is a function that agrees with $\log\log|x|^{-1}$ near the origin. Here is a derivation of a function in $\dot{W}^{1,n}(\mathbf{R}^n) \setminus L^\infty(\mathbf{R}^n)$ for $n \geq 2$. The proof reduces to finding a sequence in $\ell^n \setminus \ell^1$ (so it clearly fails for $n = 1$), and it explains the growth rate $\log\log|x|^{-1}$.

Example 1.1. Let φ be a bump function equal to 1 on $B(0,1)$ and supported in $B(0,2)$. Define

$$f(x) = \sum_{k \geq 0} a_k \varphi(2^k x) \tag{1}$$

for $a_k \geq 0$ to be chosen. Then we want

$$\|f\|_{L^\infty(\mathbf{R}^n)} = \sum_k a_k = \infty \tag{2}$$

and

$$\|f\|_{\dot{W}^{1,n}(\mathbf{R}^n)} \sim \sum_k a_k^n < \infty. \tag{3}$$

Note that when $n = 1$, the two conditions are incompatible.

We explain the second estimate. Since $\nabla(\varphi(2^k\cdot))$ have disjoint support and the $\dot{W}^{1,n}$ norm is scaling invariant, we have

$$\|f\|^n_{\dot{W}^{1,n}(\mathbf{R}^n)} = \|\sum_k a_k \varphi(2^k\cdot)\|^n_{\dot{W}^{1,n}(\mathbf{R}^n)} = \sum_k a_k^n \|\varphi(2^k\cdot)\|^n_{\dot{W}^{1,n}(\mathbf{R}^n)} = \|\varphi\|^n_{\dot{W}^{1,n}} \sum_k a_k^n. \tag{4}$$

To see where $\log\log|x|^{-1}$ comes from, take $a_k = 1/k$. Then when $|x| \sim 2^{-k}$, we have $k \sim \log|x|^{-1}$, and hence

$$f(x) \sim \sum_{j \leq k} a_k \sim \log k \sim \log\log|x|^{-1}. \tag{5}$$

2. Failure of L^2 embedding in one dimension

The L^2 version of the embedding in one dimension, namely $\dot{H}^{1/2}(\mathbf{R}) \hookrightarrow L^\infty(\mathbf{R})$, fails. Indeed, a function agreeing with $\log\log|x|^{-1}$ near the origin is in $\dot{H}^{1/2}(\mathbf{R})$ by the following trace lemma.

Lemma 2.1. *Let $u \in \mathcal{S}(\mathbf{R}^n)$ and let $s > 1/2$. Write $\widetilde{u}$ for the restriction of u to a hyperplane. We have*

$$\|\widetilde{u}\|_{H^{s-1/2}(\mathbf{R}^{n-1})} \lesssim_{n,s} \|u\|_{H^s(\mathbf{R}^n)}. \tag{6}$$

Another, more direct, explanation is as follows.

Example 2.2. We will construct functions unbounded in $L^\infty(\mathbf{R})$ but bounded in $\dot{H}^{1/2}$. For $N \geq 4$, define

$$f_N(x) = \chi_N(x)\frac{1}{x\log x} \in C_c^\infty(\mathbf{R}) \tag{7}$$

for χ_N a cutoff function equal to 1 on $[e, N]$. We have $\mathcal{F}^{-1}f_N \in L^1$, and Fourier inversion gives

$$\|\mathcal{F}^{-1}f_N\|_{L^\infty(\mathbf{R})} \geq \mathcal{F}^{-1}f_N(0) = \int_{\mathbf{R}} f_N = \log\log N + O(1). \tag{8}$$

On the other hand, the functions

$$|x|f_N^2 = \chi_N(x)\frac{1}{x\log^2 x} \tag{9}$$

are uniformly bounded in $L^1(\mathbf{R})$, so the $\mathcal{F}^{-1}f_N$ are bounded in $\dot{H}^{1/2}$.

Date: August 18, 2024.

[1]When $n = 1$, the embedding holds by the fundamental theorem of calculus.