Mass inflation for spherically symmetric charged black holes

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January 20, 2026

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General relativity

A *spacetime* is a 4-manifold $\mathcal{M}^{3+1}$ with a Lorentzian metric g solving the Einstein equations:

$$\mathrm{Ric}(g) - \frac{1}{2}R(g)g = T,$$

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Example

Minkowski space: $\mathcal{M} = \mathbf{R}^{3+1}_{t,x,y,z}$, $T = 0$ and

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

Causality

Causal character of tangent vectors

We say $v \in T_p\mathcal{M}$ is

- *spacelike* if $g(v, v) > 0$
- *timelike* if $g(v, v) < 0$
- *null* if $g(v, v) = 0$

Curves with timelike or null tangent vector define *causality*.

Initial value formulation of Einstein's equations

$$\mathrm{Ric}(g) - \frac{1}{2}R(g)g = T,$$

In the right coordinates, the Einstein equations are *quasilinear wave equations* for the metric g and the matter fields φ:

$$\begin{cases} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \mathcal{N}(g, \partial g) = \text{terms involving } \varphi \\ \text{equations for } \varphi \end{cases}$$

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Thm (Choquet-Bruhat '52, Choquet-Bruhat–Geroch '69)

Any Cauchy data set $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi})$ for the Einstein equations coupled to a suitable matter model induces a unique *maximal* "globally hyperbolic" development

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Example

Minkowski space is the unique solution arising from the data $(\mathbf{R}^3, \delta, 0, 0)$. It is *geodesically complete*, hence inextendible.

What is a black hole?

A *black hole* is a region of spacetime that “cannot be seen” by “far away observers.”

All light cones in the black hole region “point inwards.”

The past boundary $\mathcal{H}$ of the black hole region is called the *event horizon*.

The Reissner–Nordström metric

This metric describes a spherically symmetric charged black hole with mass M and charge $\mathbf{e}$:

$$g_{M,\mathbf{e}} = -\Big(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2}\Big)\, dt^2 + \Big(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2}\Big)^{-1} dr^2 + r^2 g_{\mathbf{S}^2},$$

It has an *event horizon* $\mathcal{H}$ at $r_+ = M + \sqrt{M^2 - \mathbf{e}^2}$ and a *Cauchy horizon* $\mathcal{CH}$ at $r = M - \sqrt{M^2 - \mathbf{e}^2}$.

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Important fact

The Cauchy horizon of Reissner–Nordström is *smooth*!

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Conjecture

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The matter model

We study the Einstein–Maxwell–(uncharged) scalar field system:

$$\begin{cases} \mathrm{Ric}(g) - \frac{1}{2} g R(g) = 2(T^{(\mathrm{sf})} + T^{(\mathrm{em})}), \\ T^{(\mathrm{sf})}_{\alpha\beta} = \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} g_{\alpha\beta} \partial^\mu \varphi \partial_\mu \varphi, \\ T^{(\mathrm{em})}_{\alpha\beta} = F_\alpha{}^\nu F_{\beta\nu} - \frac{1}{4} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu}, \\ \Box_g \varphi = 0, \qquad dF = 0, \qquad \mathrm{div}_g F = 0. \end{cases}$$

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We work entirely in *spherical symmetry*, where

$$g = -\Omega^2 \, du \, dv + r^2 g_{\mathbf{S}^2}, \quad F = \frac{\Omega^2 \mathbf{e}}{2r^2} du \wedge dv \ (\mathbf{e} \in \mathbf{R}).$$

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On Reissner–Nordström, the (renormalized) *Hawking mass*

$$\varpi = \frac{r}{2}\Big(1 + \frac{4 \partial_u r \partial_v r}{\Omega^2}\Big) + \frac{\mathbf{e}^2}{2r}$$

is constant and equal to the black hole mass M.

The Hawking mass controls the curvature

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Important fact

We have (when $r \geq r_0 > 0$)

$$\mathrm{Riem}^{\alpha\beta\gamma\delta}\mathrm{Riem}_{\alpha\beta\gamma\delta} \gtrsim \varpi + O(1)$$

A global existence theorem

Theorem (Dafermos ’05, ’14; Kommemi ’13)

Suitable Cauchy data for the spherically symmetric Einstein–Maxwell–scalar field system leads to a global solution containing a black hole region.

Instability of the Cauchy horizon

Linear effects (on Reissner–Nordström):

- Infinite blueshift effect at $\mathcal{CH}$ [Penrose '68, Simpson–Penrose '73, McNamara '78, Chandrasekhar–Hartle '82]

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- Einstein–null dust with one dust [Hiscock '81]
- *mass inflation* with two dusts: $\varpi|_{\mathcal{CH}} \equiv \infty$ [Poisson–Israel '89, '90; Ori '91]

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- inextendible in C^2 [Luk–Oh '19] and in $C^{0,1}_{\text{loc}}$ for small data [Sbierski '20]

A heuristic

Decay rates in exterior $\rightsquigarrow$ (in)stability results in the interior.

Best known results in the exterior

We know the pointwise upper bounds [Dafermos–Rodnianski '05]

$$|\varphi||_{\mathcal{H}} + |\partial_v \varphi||_{\mathcal{H}} \lesssim_{\epsilon,\varphi} v^{-3+\epsilon}, \tag{1}$$

and the generic L^2 lower bound [Luk–Oh '19]

$$\int_{\mathcal{H}} v^{7+\epsilon} (\partial_v \varphi)^2 \, dv = \infty \text{ for all } \epsilon > 0. \tag{2}$$

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Mass inflation generically holds if

we have (1) and a pointwise lower bound [Dafermos '05]:

$$|\partial_v \varphi||_{\mathcal{H}}(v) \gtrsim v^{-9+\epsilon},$$

or (2) and L^2 upper bounds [Luk–Oh–Shlapentokh-Rothman '22]:

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Theorem (G. '24)

All solutions satisfy

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Corollary

Mass inflation holds for generic solutions.

Price's law

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At late times, linear waves on subextremal black hole spacetimes behave like t^{-3}.

[Price '72; Dafermos-Rodnianski '05; Tataru '13; Donninger-Schlag-Soffer '12; Metcalfe-Tataru-Tohaneanu '12; Angelopoulos-Aretakis-Gajic '18, '21; Hintz '20; and many others...]

Price's law in a nonlinear and spherically symmetric setting

Theorem (Luk–Oh '19, Luk–Oh '24, G. '24)

There are constants $C_k \neq 0$, a functional $\mathfrak{L}[\varphi]$, and a small constant $\delta > 0$ such that

$$|\partial_v^k \varphi - C_k \mathfrak{L}[\varphi] v^{-3-k}||_{\mathcal{H}} \lesssim v^{-3-k-\delta} \text{ for } 0 \leq k \leq 2.$$

The quantity $\mathfrak{L}[\varphi]$ is non-zero for generic solutions.

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An application

Theorem (Van de Moortel ’25)

There exist (two-ended and spherically symmetric) asymptotically flat black holes whose interior contains both a spacelike and a null singularity.

An outline of the proof

The scaling vector field

- On Minkowski, $S_\mathrm{m} = u\partial_u + v\partial_v$ satisfies $[\Box_\mathrm{m}, S_\mathrm{m}] = 2\Box_\mathrm{m}$

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To get $|(v\partial_v)^k\varphi||_{\mathcal{H}} \lesssim v^{-1+\epsilon}$, control $|S^k\varphi|$ for $S|_{\mathcal{H}} \sim v\partial_v$.

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Key ingredients of the proof

- Redshift effect on a subextremal black hole [Dafermos–Rodnianski '05]
- Energy decay (and pointwise decay) from r^p-weighted energy estimates [Dafermos–Rodnianski '09]
- Reductive structure in the error terms arising from commutation
- Hierarchy of weak and strong decay estimates for the geometry

Reductive structure in the errors arising from commutation

$$E[\psi](\tau_2)+\iint r^{-1-\epsilon}(\partial\psi)^2 \lesssim E[\psi](\tau_1)+\iint wU\psi\Box\psi+\cdots (w>0)$$

We use three vector field commutators: U, V, and S.

- Energy estimate for φ closes on its own
- Energy estimate for $U\varphi$ sees errors involving φ
- Energy estimate for $V\varphi$ sees errors involving φ and $U\varphi$
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Takeaway

Order the commutators $U < V < S$!

Hierarchy in the estimates for the geometry

- After commuting with Γ, derive (strong) *time decay* for $\Gamma\mathfrak{g}$.
- Commuting with S requires (weak) *boundedness and r-decay* of $S\mathfrak{g}$!
- Write $|S\mathfrak{g}| \lesssim u|U\mathfrak{g}| + v|V\mathfrak{g}|$ and use time decay for $U\mathfrak{g}$ and $V\mathfrak{g}$.

The gauge

The ingoing coordinate u is normalized at null infinity:
$(-\partial_u r)|_{\mathcal{I}} = 1.$

The outgoing coordinate v is normalized on a curve of constant r:
$\partial_v r|_{\{r=r_{\mathcal{H}}\}} = 1.$

The three vector field commutators

- $\partial = (u, v)$ (used globally)
- $\overline{\partial} = (u, r)$ (used near infinity)
- $\underline{\partial} = (v, r)$ (used near the horizon)

$$U := \frac{1}{(-\partial_u r)}\partial_u, \qquad V := \chi_{r \lesssim R}(r)\underline{\partial}_v + (1 - \chi_{r \lesssim R}(r))\overline{\partial}_r,$$
$$S := \chi_{r \lesssim R}(r) v \underline{\partial}_v + (1 - \chi_{r \lesssim R}(r))(u\overline{\partial}_u + r\overline{\partial}_r).$$

Reductive structure: the details

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The global redshift vector field U

$\Box U\varphi = -\kappa UU\varphi + O(r^{-2})\partial\varphi$, where $\kappa \geq 0$

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$$E[\psi](\tau_2) + \iint r^{-1-\epsilon}(\partial\psi)^2 \lesssim E[\psi](\tau_1) + \iint wU\psi\Box\psi + \cdots \quad (w > 0).$$

The global redshift vector field U

$\Box U\varphi = -\kappa UU\varphi + O(r^{-2})\partial\varphi$, where $\kappa \geq 0$

Reductive structure: the details

$$U := \frac{1}{(-\partial_u r)}\partial_u, \qquad V := \chi_{r\lesssim R}(r)\underline{\partial}_v + (1-\chi_{r\lesssim R}(r))\overline{\partial}_r,$$

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The timelike-outgoing vector field V

- $\Box\underline{\partial}_v\varphi = O(r^{-1})\partial U\varphi + O(r^{-2})\partial\varphi$

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The timelike-outgoing vector field V

- $\Box\underline{\partial}_v\varphi = O(r^{-1})\partial U\varphi + O(r^{-2})\partial\varphi$
- $\Box\overline{\partial}_r\varphi = O(r^{-2})\overline{\partial}_r^2\varphi + O(r^{-2})\partial\varphi$

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The timelike-outgoing vector field V

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The timelike-outgoing vector field V

- $\Box\underline{\partial}_v\varphi = O(r^{-1})\partial U\varphi + O(r^{-2})\partial\varphi$
- $\Box\overline{\partial}_r\varphi = O(r^{-2})\overline{\partial}_r^2\varphi + O(r^{-2})\partial\varphi$
- $\Box V\varphi = \mathbf{1}_{r\geq R}O(r^{-2})\partial V\varphi + O(r^{-2})[\partial U\varphi + \partial\varphi]$

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The scaling vector field S

- $\Box S\varphi = O(r^{-1+\epsilon})\overline{\partial}_r^2\varphi + \text{l.o.t.}$

Reductive structure: the details

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- $\Box S\varphi = O(r^{-1+\epsilon})\overline{\partial}_r^2\varphi + \text{l.o.t.}$
- Rewrite $\overline{\partial}_r^2\varphi = r^{-1}\overline{\partial}_r(rV\varphi) + \text{l.o.t.}$

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$$U := \frac{1}{(-\partial_u r)}\partial_u, \qquad V := \chi_{r\lesssim R}(r)\underline{\partial}_v + (1-\chi_{r\lesssim R}(r))\overline{\partial}_r,$$

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- $\Box S\varphi = O(r^{-1+\epsilon})\overline{\partial}_r^2\varphi + \text{l.o.t.}$
- Rewrite $\overline{\partial}_r^2\varphi = r^{-1}\overline{\partial}_r(rV\varphi) + \text{l.o.t.}$
- $\Box S\varphi = O(r^{-2+\epsilon})\overline{\partial}_r(rV\varphi) + \text{l.o.t.}$

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$$E_p[V\varphi](\tau_2) + \iint r^{-3+3\epsilon}(\overline{\partial}_r(rV\varphi))^2 \lesssim E_p[V\varphi](\tau_1), \quad (p=3\epsilon).$$

Summary of the proof

- Construct a scaling vector field commutator S with $S|_{\mathcal{H}} \sim v\partial_v$
- Introduce vector field commutators U and V so that U, V, and S exhibit a reductive structure when $U < V < S$
- To close the energy estimate for $\Gamma\varphi$, use the reductive structure and (weak) boundedness and r-decay for $\Gamma\mathfrak{g}$ obtained using the (strong) time decay for $\Gamma'\mathfrak{g}$ with $\Gamma' < \Gamma$
- Obtain (strong) time decay for $\Gamma\varphi$
- Deduce $v^{-1+\epsilon}$ decay for $\Gamma\varphi$ using standard techniques
- Take $\Gamma = S^k$ for k large and use known results to obtain mass inflation