ONYX GAUTAM'S GENERALS

Committee: Igor Rodnianski [R], Mihalis Dafermos [D], and Jakub Witaszek [W].
Special topics: harmonic analysis and partial differential equations
Date: 27 September 2024
Time: 12:15–14:45
Location: Fine 300 (the chair's office)

I arrive at 12:15 to [R]'s office, a little worried about being the last one there. [D] and [W] are not there yet, so [R] and I chat about how this is my first ever oral exam and how in America, most exams are written, while in St Petersburg (where [R] did his undergrad), all exams were oral. [W] arrives, and we chat about the painting of Walter Minto in the chair's office and the prizes (the Nobel Prize of Wigner and the Abel Prize of Sinai) that are in the closet. [D] finally arrives and we begin the exam.

1. Algebra

[W]: Can a group of order 10 act on a set of size 9 such that there are exactly two orbits?

(I am quite nervous and have no idea where to start. I say that the sizes of the orbits have to divide 9.)

[W]: Do you know any theorems about orbits of group actions?

(I state the orbit stabilizer theorem and realize that the size of the orbits must divide 10, the order of the group.)

[W]: Can you define the words "orbit" and "stabilizer"?

(I do so, but get stuck on the rest of the question.)

[W]: Where do the orbits live?
[G]: Ah, they are a subset of S, and they have to be disjoint, so this cannot happen, since the divisors of 10 are 1, 2, 5, and 10, and no two of them sum up to 9.
[W]: Can you define a PID?
[G]: A domain where every ideal is generated by one element.
[W]: Can you define a domain?
[G]: A ring where multiplication by a non-zero element is injective.
[W]: Do you know any theorems about modules over a PID?

(State the structure theorem for finitely generated modules over a PID, and mention primary decomposition and invariant factor decomposition.)

[W]: What does this theorem have to do with the Jordan canonical form of a matrix?

(I explain how, over an algebraically closed field, the Jordan canonical form comes from the primary factor decomposition.)

[W]: Can you tell me anything about a domain that is a finite dimensional vector space over a field?
[G]: (After some time) It is the field itself?
[W]: Is that true?
[G]: Oh, $\mathbf{C}$ over $\mathbf{R}$. Maybe it's always a field.
[W]: Can you prove it?
[G]: (After some more thought) I need to prove that multiplication by a non-zero element is surjective, but by linear algebra this is the same as proving that such a map is injective, and that holds by the domain property.

([R] asks [D] if he wanted to ask any algebra, and [D] jokes that he forgot all his algebra shortly after being taught it at Harvard. I ask if we could do complex analysis next.)

2. Complex analysis

[D]: I'm going to ask a question that might not be complex analysis—it might be real analysis, or it might be PDE—because I don't like the usual complex analysis questions. Consider the Laplace equation.

(I write the symbol Δ.)

[R]: The equation, not the operator!

(I write $\Delta u = 0$).

[D]: Let's say we're in $\mathbf{R}^2$, and I give you real analytic data for u and $\partial_y u$ on the x-axis. Can you solve this Cauchy problem locally, in a neighbourhood of the x-axis?

[G]: This reminds me of Cauchy–Kovalevskaya.

(I just reviewed the statement with Tuomas and then with Igor, but I am still nervous because I am not that comfortable with this theorem.)

[R]: Can you spell Kovalevskaya?

(I do.)

[R/D]: A Russian mathematician!

[R]: State the theorem.

[G]: I will just state the theorem generally, and specialize it to our case later.

(I write down $\sum_{|\alpha|=k} F_\alpha(\partial^{<k}\varphi, x)\partial^\alpha\varphi + a(\partial^{<k}\varphi, x) = 0.$)

[G]: Given real analytic data on the x-axis, there is a real analytic solution in a neighbourhood of the x-axis, as long as F_{yy} doesn't vanish on the x-axis.

[D]: What if instead of the x-axis we had a general hypersurface?

[G]: Write the PDE in coordinates where the first $k-1$ are tangent to the hypersurface and the k-th is normal to the hypersurface. The theorem works as long as the term in front of the k-th normal derivative doesn't vanish.

[D]: What is a more geometric way of saying this?

(I get confused with my multi-index notation and [R] tells me to just set $k = 2$ and suppose the coefficients of the PDE depend only on x, and to forget about the lower order term a. I write out $F_{xx}(x)\partial_x^2\varphi + F_{xy}(x)\partial_{xy}\varphi + F_{yy}(x)\partial_y^2\varphi$ and say something about the F_{yy} term again.)

[D]: But can you phrase that more geometrically? What property does the normal vector need to have?

(I still don't get it.)

[R]: This expression you have written down looks like a metric.

[D]: Well, let's not call it a metric.

[G]: It's a form.

[D]: Which you can suppose is symmetric.

[G]: OK, so the the normal vector needs to not have "inner product" zero with itself.

[D]: Let's go back to the original problem.

[G]: We are done by Cauchy–Kovalevskaya.

[D]: Now let's suppose the data is only smooth. Can you still solve this problem?

[G]: My guess is no. (After some thought) Harmonic functions are analytic, so the data would be analytic a posteriori.

[D]: We'll get back to how you prove that in a second. Let's say you have a solution with smooth data to the "future" ($y \geq 0$). Can you also produce a solution to the past?

[G]: I thought we just said the data had to be analytic for this to work.

[D]: The solution could immediately become analytic to the future, like for the heat equation.

(I don't know what to do.)

[D]: Suppose also that $\partial_y u = 0$ on the x-axis. Can you produce a past solution then?

[G]: Even reflection.

[D]: Why does that work?

[G]: It solves the equation in $\{y < 0\}$, and it is C^2 across the x-axis, so it solves the equation everywhere. (Getting confused) But the first derivative isn't continuous...

[D]: It's OK because $\partial_y u = 0$ on the x-axis. Now how do you prove that harmonic functions are analytic?

[G]: It's a local problem, so I can prove it on balls, and there I can produce a harmonic conjugate.

([D] asks me to say a few words about that, and I give a proof of existence of the harmonic conjugate. [W] asks some clarifying questions.)

[D]: And if we are in higher dimensions? Then how do you prove harmonic functions are analytic?
[G]: Write the mean value formula in a ball, differentiate under the integral sign a bunch of times to get gradient estimates, then show that the Taylor series converges.
[D]: Is that enough to be analytic?
[G]: (Realizing there are non-analytic functions whose Taylor series converge) No, you need some uniform bounds on the derivative, but we have those.

3. Real analysis

[R]: Let's do some more Russian theorems.
[D]: (to [R]) If we want a Russian theorem, you just need to prove something.

([D] and [R] laugh).

[R]: State Egorov's theorem.

(I panic because I did not study measure theory. I start talking about sets where measurable functions are continuous, and I am told this is Lusin's theorem. I then state the correct theorem.)

[D]: For my intuition, can you give me an example where the set of uniform convergence is not just the whole domain? Let's say we're on the unit interval.

(After a silent minute, I say that what I am trying to do is to come up with an example where the functions converge to zero but stay big at some places, to prevent uniform convergence. I come up with $\mathbf{1}_{[0,1/n]}$).

[R]: (to [D]) Is that enough for your intuition?

([R] and [D] laugh.)

[R]: Vague idea of a proof?
[G]: (Knowing that I definitely did not review measure theory. . .) Maybe I could do it with a hint?
[R]: Well, it's the usual thing. First you have to come up with the set. . .

(I say some words about how these types of measure theory theorems are proved.)

[R]: As far as vague ideas go, that's OK.
[D]: Let's do another Russian theorem. Do you know Chebyshev's inequality?

(I state it. [D] says that since the proof is not hard, they will not ask me about it.)

4. Harmonic analysis

([R] tells [W] that he is free to leave for the special topics.)

[R]: State the Mikhlin multiplier theorem.

(I do.)

[R]: How do you prove this theorem?
[G]: You show that the physical space kernel associated to the Mikhlin multiplier is a Calderón–Zygmund kernel. Should I try to give a proof? (I know this proof very well.)
[R]: (Shakes head.) What is a Calderón–Zygmund kernel?

(I write down the size condition $|K(x)| \lesssim |x|^{-d}$, the L^2-boundedness condition $|\hat{K}| \lesssim 1$, as well as the regularity condition $\int_{|x|\geq 2|y|} |K(x) - K(x-y)| \, \mathrm{d}x \lesssim 1$. I mention that in practice one checks the third condition by showing that $|\nabla K(x)| \lesssim |x|^{-d-1}$.)

[R]: How do you define an operator from this kernel?

(I write down $T_\epsilon f(x) := \int_{|y|\geq\epsilon} K(y) f(x-y) \, \mathrm{d}y$ and $Tf := \lim_{\epsilon\to 0} T_\epsilon f$ and say that this defines an operator for sufficiently smooth and decaying functions.)

[R]: What estimates do you have for such operators?
[G]: They are $L^p \to L^p$ for all $p \in (1, \infty)$.
[R]: How do you prove that?

[G]: The class of such kernels is closed under taking formal adjoint, because the adjoint of the operator with kernel $K(x)$ has kernel $\overline{K}(-x)$, so by duality it is enough to obtain the exponents $p \in (1,2]$. It's L^2 bounded by assumption, so by Marcinkiewicz interpolation I need a weak $(1,1)$ estimate.
[R]: And how do you prove that?

(I start to give the proof. By the time I finish stating the properties of the Calderón–Zygmund decomposition of an L^1 function, [R] stops me and says he believes me that I can give the rest of the proof.)

[D]: But what is the Mikhlin multiplier theorem good for? I'm not a harmonic analyst. What's an example of a multiplier *I* would care about?
[G]: A compactly supported multiplier.
[D]: No, you can't say that.
[G]: You don't like projecting to low frequencies?

(We all laugh.)

[G]: You can use the Mikhlin multiplier theorem, together with an inequality from probability theory, to prove the Littlewood–Paley square function estimate.
[R]: Khinchin. Another Russian!

(I had no idea how to pronounce that, so I didn't say the name.)

[R]: I want something more basic than Littlewood–Paley. Consider the Poisson equation. How would you solve this with the Mikhlin multiplier theorem?

(I write down $\Delta u = f$ and take Fourier transform to get $\hat{u} = -|\xi|^{-2}\hat{f}$).

[G]: But $|\xi|^2$ is not a Mikhlin multiplier.
[R]: So Mikhlin tells you cannot control u in L^p by f in L^p.
[G]: Well, u should not be in L^p, since it grows at infinity.
[R]: Let's say we want a local estimate.
[G]: I would introduce cutoff functions. . .
[R]: Introducing cutoffs would not play nicely with your Fourier transforms—we would have to leave the world of Mikhlin multipliers. You would get a pseudodifferential operator.
[D]: What can you do with the Poisson equation to produce a Mikhlin multiplier?

(I write down $\xi_i/|\xi|$ and say that this is a Mikhlin multiplier.)

[R]: But you need $|\xi|^2$!

(I write down $\xi_i\xi_j/|\xi|^2$).

[G]: This is the multiplier for $(-\Delta)^{-1}\partial_i\partial_j$, so we can control the second derivatives of u in L^p by the Laplacian of u in L^p.
[D]: What are such estimates called?
[G]: (Not sure what he wants) Calderón–Zygmund estimates for the Laplacian.
[R]: Do you know any other ways to prove the weak $(1,1)$ estimate, without using Calderón–Zygmund decomposition?
[G]: No.
[R]: Really? Do you know the Cotlar–Stein lemma?
[G]: I've seen it, but I might have forgotten how it works.
[R]: What is the point of the lemma?
[G]: Almost orthogonality. If you have an operator that is the sum of some other operators (I write down $T = \sum_{i=1}^{N} T_j$) which are almost mutually orthogonal, in the sense that $\|T_j^*T_k\| \le \gamma(|j-k|)$ and $\|T_jT_k^*\| \le \gamma(|j-k|)$ for some summable kernel γ, then you get some estimate for the operator T.
[R]: Then you want a square root on the left side.

(I correct my mistake.)

[R]: Any idea of the proof?
[G]: You expand out $(T^*T)^n$ and estimate it in two different ways.
[R]: Why don't you just take $n = 1$?
[G]: Because you need the constant to be uniform in n.
[R]: Which n are you talking about?

[G]: Big N! You want the constant in the estimate to be independent of N, the number of terms in the sum, so you look at $(T^*T)^n$ and take n-th roots.
[R]: How does Cotlar–Stein help you prove something about Calderón–Zygmund operators?
[G]: You break the integral into dyadic blocks and show that the blocks are mutually orthogonal.
[R]: Can you show me why a dyadic block is bounded on L^p?

(I have no idea.)

[R]: Do you know any theorems that tell you when an integral operator is bounded on L^p?
[G]: No idea.
[R]: You are being too quick to say you don't know. How do you control the size of a convolution?
[G]: There's Young's inequality.
[R]: State it.

(I write $\|f * g\|_r \lesssim \|f\|_p \|g\|_q$ for $1 + 1/r = 1/p + 1/q$.)

[R]: So if you want q and r to be equal...
[G]: (I get it now.) You need the kernel to be in L^1. Which is true for a dyadic block because of the size bound $|K| \lesssim |x|^{-n}$.
[R]: Do you know anything about Strichartz estimates?
[G]: Not enough to say here.
[R]: Do you know what they have to do with Fourier restriction?
[G]: Strichartz estimates for the Schrödinger equation have to do with restriction theorems for the paraboloid, and for the wave equation you need to look at a cone.
[R]: But?

(I have no idea.)

[R]: You don't get all the Strichartz estimates this way. The restriction estimates are isotropic, so you only get the Strichartz estimates where the exponents in space and time are equal.
[R]: What do you know about pseudodifferential operators?
[G]: I know nothing about pseudodifferential operators.
[R]: Then we'll do it by hand. Consider the following operator: first cutoff in physical space to the ball of radius 1, then apply a Mikhlin multiplier, then cutoff again in physical space, to a ball disjoint from the ball of radius 1.

(I write down $Tf = \psi \mathcal{F}^{-1} m \mathcal{F}(\chi f)$).

[R]: You can draw a picture if you want.

(I draw two disjoint circles.)

[R]: Let's say $f \in L^2$. What can you tell me about Tf?
[G]: Cutting off keeps you in L^2, and so does the multiplier, so $Tf \in L^2$.
[R]: What can you tell me about derivatives of Tf?

(I compute $Tf = \psi m^\vee * (\chi f)$ and start trying to write the derivative in the Fourier side.)

[R]: Write it out without using convolutions.
[G]: What do you mean?
[R]: Fourier transforms are integrals, write it out like that.

(I compute $Tf(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \psi(x) e^{ix\cdot\xi} m(\xi) e^{-iy\cdot\xi} \chi(y) f(y) \, \mathrm{d}y \, \mathrm{d}\xi$.)

[R]: Can you differentiate this?

(I write $\partial_j Tf(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \psi(x) i\xi_j e^{ix\cdot\xi} m(\xi) e^{-iy\cdot\xi} \chi(y) f(y) \, \mathrm{d}y \, \mathrm{d}\xi$.)

[G]: This looks bad because $m(\xi)\xi_j$ grows. I want to integrate by parts in y, but it could hit on f, and I don't have information about derivatives of f.
[R]: There's another way you can integrate by parts. Combine the exponentials.

(I write $\partial_j Tf(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \psi(x) i\xi_j m(\xi) e^{i(x-y)\cdot\xi} \chi(y) f(y) \, \mathrm{d}y \, \mathrm{d}\xi$.)

[R]: How can you use the fact that ψ and χ have disjoint supports?
[G]: The difference $x-y$ is bounded below, so I can write $\partial_j Tf(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{1}{x_j - y_j} \psi(x)\chi(y)\xi_j m(\xi) \partial_{\xi_j} e^{i(x-y)\cdot\xi} f(y) \, \mathrm{d}y \, \mathrm{d}\xi$. Then I integrate by parts in ξ. It can hit on m or on ξ_j, but either way I lose a power of ξ.
[R]: So derivatives of Tf are in L^2. What about higher derivatives?
[G]: I can do this again, and integrate by parts as many times as I want.

[R]: Can you really do it as many times as you want?
[G]: Oh, if it hits on m too many times, then it will not be integrable near the origin.
[R]: How can you get around this?
[G]: Cut off around the origin.
[R]: Exactly, you split the integral into high frequencies and low frequencies—(to [D]) since Mihalis likes low frequencies—and only integrate by parts in the high frequency region.
[G]: And for low frequencies there is no problem with ξ-growth, since ξ is bounded.
[R]: Right. What would you guess that these operators are called?
[G]: Pseudodifferential operators of order $-\infty$?
[R]: Close enough. They are called smoothing operators.

5. PDE

[D]: Consider a 2-sphere in Minkowski space together with its future ingoing and outgoing light cones. I give you real analytic initial data for the wave equation on the cones. Can you produce a local solution?
[R]: First of all, what type of data should you prescribe?
[G]: The function itself.

([R] approves.)

[G]: I can't use Cauchy–Kovalevskaya...
[D]: But can you solve it using the Cauchy–Kovalevskaya idea?

(I pick a point on the cone (not on the sphere) and start trying to write down a power series expansion for the solution.)

[G]: I am confused about what coordinate system to use.
[D]: As far as I am concerned there is only one coordinate system to use!

(I write a power series in double null coordinates, and get confused.)

[G]: It seems I have picked the wrong point.

(I instead choose a point on the 2-sphere and centre my power series at that point.)

[D]: Let's say we're in $2+1$ dimensions, so you can draw it better.

(I start trying to compute the wave equation in (2+1)-dimensional polar coordinates. I write down $x = r\cos\theta$ and $y = r\sin\theta$, realize that I don't want to do this computation in public, and erase what I have written.)

[G]: Can you tell me the form of the wave equation in this dimension and in these coordinates?

([D] tells me the answer, and he and [R] have a brief discussion about how you can figure out some of the terms without any computation.)
(I write a power series expansion in u, v, and the angular variable φ. I feel weird writing φ^k where φ is an angle.)

[D]: Maybe you should write out the first few terms of this power series.

(I very slowly and painfully do so, and realize that the ∂_u, ∂_v, and ∂_φ terms are determined from initial data. [R] makes the joke that by induction all the terms are determined. I then realize that the ∂_u^2, ∂_v^2, and ∂_φ^2 terms are determined from the data, and so are the cross terms $\partial_u\partial_\varphi$ and $\partial_v\partial_\varphi$. On the other hand, the $\partial_u\partial_v$ term is determined from the equation. Throughout this process, they are making sure to ask me why these quantities are determined.)

[G]: The only terms not immediately determined from data are terms with mixed u and v derivatives, but for those I can use the equation. Then I would need to carefully track some estimates on the size of the derivatives to make sure this power series converges.
[D]: Let's say you can do that. What does this tell you?
[G]: There is an analytic solution in a neighbourhood of the initial circle.
[R]: Do you know Holmgren's uniqueness theorem?
[G]: No.
[R]: It's OK. Holmgren, Hörmander... for today you don't need to know them—only Russian mathematicians!
[D]: What if I ask the same problem, but the data I give you is only smooth?

[G]: I would establish an estimate, and approximate the smooth data by analytic data and show that some convergence holds in the topology of the norms of the estimate.
[R]: What estimate?
[G]: I would integrate the equation in u and v to get an estimate.
[R]: In higher dimensions, you cannot just integrate the equation—you need something a bit more high-powered.
[G]: Ok, I'll multiply the equation by $\partial_t \psi$ and integrate by parts in the usual way. All derivatives are controlled on the top boundary, and the tangential derivatives are controlled on the cone.
[D]: Now consider rotating the picture by 90 degrees. That is, can you solve the sideways characteristic initial value problem?

(I draw a triangular region with a timelike boundary on the right, recalling that I reviewed something like this with Igor during mock generals.)

[G]: No, and you can see it from the estimate. The boundary term on the right will have a bad sign.
[D]: A bad sign or no sign?
[G]: The angular term has a bad sign, and the other term has a good sign, so there is no sign. So this will not help me disprove existence.
[R]: What do bad and good sign mean when there is an identity? Just write down the energy flux on the boundary components.
[G]: On the cone it's the tangential derivatives, and on the timelike boundary it's $|\partial_t \psi|^2 - |\partial_\varphi \psi|^2$.
[D]: Is it really ∂_t?
[G]: (I think for a bit.) No, it's $|\partial_r \psi|^2 - |\partial_\varphi \psi|^2$.
[D]: With this argument it will be difficult to talk about existence. Let's forget about existence for now and talk about uniqueness.
[G]: So if the data is zero, I want to show the solution is zero.
[D]: Can you do it when the solution is rotationally symmetric?
[G]: Yes, the estimate gives $\int |\partial_r \psi|^2 = 0$ on the timelike boundary. I can integrate this in the r direction and conclude by a Poincaré inequality that ψ is zero.
[D]: What is a generalization of rotationally symmetric?
[G]: Spherically symmetric?
[D]: Yes, the same argument will work in higher dimensions. How would you phrase rotationally symmetric in terms of harmonic analysis?
[G]: Oh, you want me to do it mode by mode. Rotationally symmetric means that the zeroth mode vanishes.
[R]: Do it when ψ is supported on the k-th mode.

(I write $\int \psi^2 = \frac{1}{k^2} \int |\partial_r \psi|^2$.)

[R]: What does this remind you of?
[G]: A Poincaré inequality.
[D]: How does the constant in that inequality depend on the domain?
[G]: It grows linearly in the diameter of the domain.
[R]: You are using ∂_r, so it is just the width of the domain.
[G]: If the width of the domain is smaller than a constant times $1/k^2$, then Poincaré's inequality gives a contradiction, So I obtain uniqueness in a very small triangular region whose width is like $1/k^2$.
[D]: Can you obtain uniqueness in the whole region?
[G]: Yes, I repeat this argument in a trapezoidal slabs of width $\sim 1/k^2$.
[D]: A general smooth function can be decomposed into angular modes, and if each of them is zero then the function itself is zero, so you are done.

(At this point [D] and [R] look at each other and ask me if we are done. It was only 2:25, and we had until 3:00, and I did not feel very happy with my performance on the PDE questions, and I wanted something with no Cauchy–Kovalevskaya, so I ask for another question. [D] and [R] look at each other again.)

[G]: (Backtracking) Or maybe we can just stop.
[R]: I won't get in the way of someone who wants more! What equation?

(I said they could pick, hoping for anything but the Burgers' equation. [R] asks if I know Vlasov. I don't, so I ask for another wave equation problem. [R] says that would be too easy. [R] asks if I know Di Giorgi–Nash–Moser theory, and I say no.)

[R]: Ok, we can do Moser iteration. Consider the Poisson equation. What estimates can you get in physical space?

(I write down $\Delta u = f$.)

[G]: L^2 estimates.

[R]: What about L^p estimates, without using Calderón–Zygmund.

[G]: One could differentiate the Newton potential under the integral sign, but other than that I don't know.

[R]: How about multiplying the equation by a power of u?

(I write $\int u^q \Delta u = \int f u^q$.)

[G]: I want to integrate by parts, but I am worried about the boundary terms.

([R] makes the joke that you should never worry about boundary terms, except when you should. He says that one can make this all rigorous by introducing cutoff functions.)

(I integrate by parts to get $\int q u^{q-1}|Du|^2 = -\int f u^q$.)

[R]: But that looks difficult to work with. Can you write the left side as a derivative of something squared? And let's not worry about constants.

(I compute the left side to be like $\int |Du^{(q+1)/2}|^2$.)

[R]: Does this give you control over u in some L^p norm?

[G]: I could use a Poincaré inequality, but I don't know the boundary values.

[R]: What about Sobolev embedding? Let's say we are in $\mathbf{R}^3$.

[G]: Oh, $\dot{H}^1 \hookrightarrow L^6$, so I control u in L^{3q+3}. Then I should Hölder the right side.

[R]: Let's not worry about the right side. Let's say f is as nice as you want, and so the only question is what norm of f is on the right side. What would you do from here?

[G]: I would do this procedure again, since we have essentially improved q to $3q + 3$.

[R]: What's the best thing one could hope for?

[G]: That u is in L^p for all p.

[R]: And even that u is bounded.

[G]: You would need uniform control on the L^p norm of u for that.

[R]: And how would you get such control?

[G]: By carefully tracking the constants we ignored in our previous estimates and taking an appropriate root.

(I step outside. They come out, congratulate me, and comment that the deliberation was short.)

6. Remarks

I have tried to make this account as truthful as possible to the actual exam. In particular, I want to convey the informal nature of my exam. There was lots of back and forth, and it really felt like a conversation. In truth there were even more hints, jokes, and side discussions than I could represent in written form. They did not want me to present lots of details for theorems I was comfortable with, but they also did not let me stand there with no idea how to proceed for very long.

I would like to thank Kunal Chawla, Sergio Cristancho, Alper Gunes, Igor Medvedev, Tuomas Tuukkanen, and Kevin Ren for holding excellent mock generals for me. Thanks especially to Igor and Tuomas for asking me PDE questions. The practice exams really helped me build my confidence, work on my board technique, and gain experience being stuck at the board.