Mass inflation for spherically symmetric subextremal charged black holes

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Princeton University
ICERM workshop on extremal black holes

Which picture is generic?

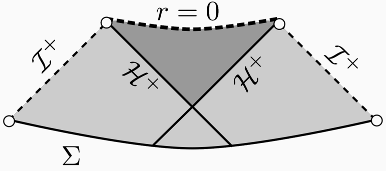

Figure 1: Schwarzschild

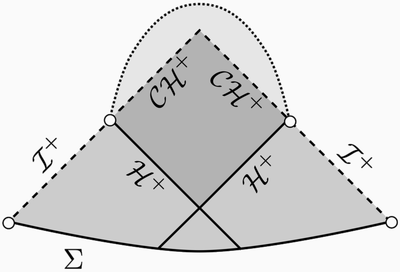

Figure 2: Subextremal Reissner–Nordström.

Figures from [Dafermos–Luk '17]

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Conjecture

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The Reissner–Nordström metric

This metric describes a spherically symmetric charged black hole with mass M and charge $\mathbf{e}$:

$$g_{M,\mathbf{e}} = -\left(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2}\right)^{-1} dr^2 + r^2 g_{\mathbf{S}^2},$$

We work entirely in the *subextremal* case $|\mathbf{e}| < M$.

The matter model

We study the Einstein–Maxwell–(uncharged) scalar field system:

$$\begin{cases} \mathrm{Ric}(g) - \frac{1}{2} g R(g) = 2(T^{(\mathrm{sf})} + T^{(\mathrm{em})}), \\ T^{(\mathrm{sf})}_{\alpha\beta} = \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} g_{\alpha\beta} \partial^\mu \varphi \partial_\mu \varphi, \\ T^{(\mathrm{em})}_{\alpha\beta} = F_\alpha{}^\nu F_{\beta\nu} - \frac{1}{4} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu}, \\ \Box_g \varphi = 0, \qquad dF = 0, \qquad \mathrm{div}_g F = 0. \end{cases}$$

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We work entirely in *spherical symmetry*, where

$$g = -\Omega^2 \, du \, dv + r^2 g_{\mathbf{S}^2}, \quad F = \frac{\Omega^2 \mathbf{e}}{2r^2} du \wedge dv \ (\mathbf{e} \in \mathbf{R}).$$

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On Reissner–Nordström, the *Hawking mass*

$$\varpi = \frac{r}{2}\Big(1 + \frac{4\partial_u r \partial_v r}{\Omega^2}\Big) + \frac{\mathbf{e}^2}{2r}$$

is constant and equal to the black hole mass M.

The a priori picture

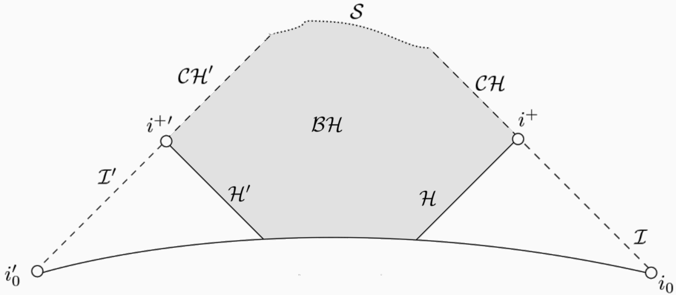

Figure 3: The a priori Penrose diagram for solutions to this model.
[Dafermos '05, Dafermos '14, Kommemi '13]

Instability of the Cauchy horizon

Linear effects (on Reissner–Nordström):

- Infinite blueshift effect at $\mathcal{CH}$ [Penrose '68, Simpson–Penrose '73, McNamara '78, Chandrasekhar–Hartle '82]

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The extremal case

Mass inflation does not occur! [Gajic–Luk '17]

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- inextendible in C^2 [Luk–Oh '19] and in $C^{0,1}_{\text{loc}}$ for small data [Sbierski '20]

A heuristic

Decay rates in exterior $\rightsquigarrow$ (in)stability results in the interior.

Best known results in the exterior

We know the pointwise upper bounds [Dafermos–Rodnianski '05]

$$|\varphi||_{\mathcal{H}} + |\partial_v \varphi||_{\mathcal{H}} \lesssim_{\epsilon,\varphi} v^{-3+\epsilon}, \tag{1}$$

and the generic L^2 lower bound [Luk–Oh '19]

$$\int_{\mathcal{H}} v^{7+\epsilon} (\partial_v \varphi)^2 \, dv = \infty \text{ for all } \epsilon > 0. \tag{2}$$

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we have (1) and a pointwise lower bound [Dafermos '05]:

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Corollary

Mass inflation holds for generic solutions.

Price's law

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At late times, linear waves on subextremal black hole spacetimes behave like t^{-3}.

[Price '72; Dafermos-Rodnianski '05; Tataru '13; Donninger-Schlag-Soffer '12; Metcalfe-Tataru-Tohaneanu '12; Angelopoulos-Aretakis-Gajic '18, '21; Hintz '20; and many others...]

Price's law in a nonlinear and spherically symmetric setting

Theorem (Luk–Oh '19, Luk–Oh '24, G. '24)

There are constants $C_k \neq 0$, a functional $\mathfrak{L}[\varphi]$, and a small $\delta > 0$, such that

$$|\partial_v^k \varphi - C_k \mathfrak{L}[\varphi] v^{-3-k}||_{\mathcal{H}} \lesssim v^{-3-k-\delta} \text{ for } 0 \leq k \leq 2.$$

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An application

Theorem (Van de Moortel '25)

There exist (two-ended and spherically symmetric) asymptotically flat black holes whose interior contains both a spacelike and a null singularity.

An outline of the proof

The region of spacetime under consideration

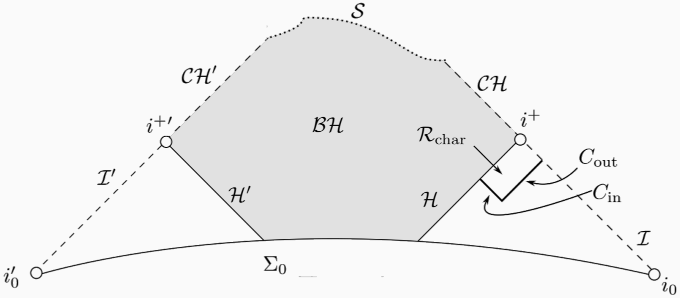

Figure 4: We work in the late-time region $\mathcal{R}_{\mathrm{char}}$.

The foliation of the exterior region

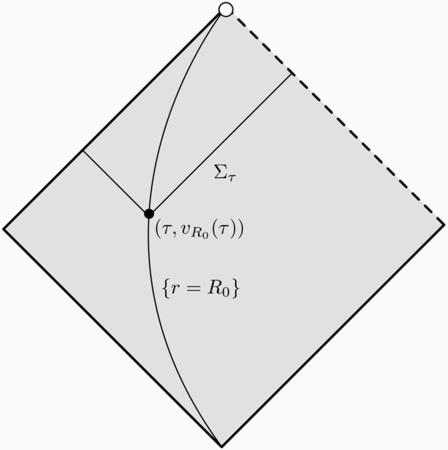

Figure 5: We foliate the exterior region (at late-times) by bifurcate null hypersurfaces Σ_τ.

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Key ingredients of the proof

- Redshift effect on a *subextremal* black hole [Dafermos–Rodnianski ’05]
- Energy decay (and pointwise decay) from r^p-weighted energy estimates [Dafermos–Rodnianski ’09]
- Reductive structure in the error terms arising from commutation
- Hierarchy of weak and strong decay estimates for the geometry

Reductive structure in the errors arising from commutation

We use three vector field commutators, U, V, and S.

- Energy estimate for φ closes on its own
- Energy estimate for $U\varphi$ sees errors involving φ
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Takeaway

Order the commutators $U < V < S$!

Hierarchy in the estimates for the geometry

- After commuting with Γ, derive (strong) *decay* for $\Gamma\mathfrak{g}$.
- Commuting with S requires (weak) *boundedness* of $S\mathfrak{g}$!
- Write $|S\mathfrak{g}| \lesssim u|U\mathfrak{g}| + v|V\mathfrak{g}|$ and use decay for $U\mathfrak{g}$ and $V\mathfrak{g}$.

The gauge

The ingoing coordinate u is normalized at null infinity:
$(-\partial_u r)|_{\mathcal{I}} = 1$.

The outgoing coordinate v is normalized on a curve of constant r:
$\partial_v r|_{\{r=r_{\mathcal{H}}\}} = 1$.

The three vector field commutators

- $\partial = (u, v)$ (used globally)
- $\overline{\partial} = (u, r)$ (used near $\mathcal{I}$)
- $\underline{\partial} = (v, r)$ (used near $\mathcal{H}$)

$$U := \frac{1}{(-\partial_u r)}\partial_u, \qquad V := \chi_{r \lesssim R}(r)\underline{\partial}_v + (1 - \chi_{r \lesssim R}(r))\overline{\partial}_r,$$

$$S := \chi_{r \lesssim R}(r) v \underline{\partial}_v + (1 - \chi_{r \lesssim R}(r))(u\overline{\partial}_u + r\overline{\partial}_r).$$

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$$E[\psi](\tau_2) + \iint r^{-1-\epsilon}(\partial\psi)^2 \lesssim E[\psi](\tau_1) + \iint wU\psi\Box\psi + \cdots \quad (w > 0).$$

The global redshift vector field U

$\Box U\varphi = -\kappa\partial U\varphi + O(r^{-2})\partial\varphi$, where $\kappa \geq 0$

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- $\Box\underline{\partial}_v\varphi = O(r^{-1})\partial U\varphi + O(r^{-2})\partial\varphi$
- $\Box\overline{\partial}_r\varphi = O(r^{-2})\overline{\partial}_r^2\varphi + O(r^{-2})\partial\varphi$
- $\Box V\varphi = \mathbf{1}_{r\geq R}O(r^{-2})\partial V\varphi + O(r^{-2})[\partial U\varphi + \partial\varphi]$

Reductive structure: the details

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- $\Box S\varphi = O(r^{-1+\epsilon})\overline{\partial}_r^2\varphi + \text{l.o.t.}$

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$$E_p[V\varphi](\tau_2) + \iint r^{-3+3\epsilon}(\overline{\partial}_r(rV\varphi))^2 \lesssim E_p[V\varphi](\tau_1), \quad (p = 3\epsilon).$$