NOTES ON THE KALUZA–KLEIN REDUCTION

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This is an expanded version of the calculation in [Cic25, Prop. 2.3].

Proposition 1 (Kaluza–Klein reduction)**.** *Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension $n \geq 3$, and let φ be a scalar field on $\mathcal{M}$. Equip $\tilde{\mathcal{M}} \coloneqq \mathcal{M} \times S^1$ with the Lorentzian metric*

$$\tilde{g} \coloneqq g + e^{2A\varphi}\,\mathrm{d}\theta^2. \tag{1}$$

If $A = 2/\sqrt{3}$ and $h \coloneqq e^{2B\varphi} g$ for $B = A/(n-2)$, then $(\tilde{\mathcal{M}}, \tilde{g})$ solves the Einstein vacuum equations if and only if $(\mathcal{M}, h, \varphi)$ solves the Einstein–scalar field equations:

$$\mathrm{Ric}[\tilde{g}] = 0 \iff \begin{cases} \mathrm{Ric}[h]_{ij} = 2\partial_i\varphi\partial_j\varphi \\ \Box_h\varphi = 0. \end{cases} \tag{2}$$

We first compute the Ricci curvature of $\tilde{g}$. We use Greek indices on $\tilde{\mathcal{M}}$, Latin indices on $\mathcal{M}$, and the index 5 on S^1.

Lemma 2. *The non-zero Christoffel symbols of $\tilde{g}$ are*

$$\Gamma[\tilde{g}]^k_{ij} = \Gamma[g]^k_{ij}, \qquad \Gamma[\tilde{g}]^k_{55} = -Ae^{2A\varphi} g^{kj}\partial_j\varphi, \qquad \Gamma[g]^5_{5j} = A\partial_j\varphi. \tag{3}$$

Proof. Straightforward calculation. □

Lemma 3. *The Ricci curvature of $\tilde{g}$ is given by*

$$\mathrm{Ric}[\tilde{g}]_{ij} = \mathrm{Ric}[g]_{ij} - A\nabla_i\nabla_j\varphi - A^2\partial_i\varphi\partial_j\varphi, \qquad \mathrm{Ric}[\tilde{g}]_{5j} = 0, \qquad \mathrm{Ric}[\tilde{g}]_{55} = -Ae^{2A\varphi}(\Box_g\varphi + Ag^{kj}\partial_k\varphi\partial_j\varphi), \tag{4}$$

where ∇ is the Levi–Civita connection of g.

Proof. Using lemma 2, we compute

$$\mathrm{Ric}[\tilde{g}]_{5j} = 0 \tag{5}$$

and

$$\begin{aligned} \mathrm{Ric}[\tilde{g}]_{55} &= \partial_\alpha\Gamma[\tilde{g}]^\alpha_{55} - \cancel{\partial_5\Gamma[\tilde{g}]^\alpha_{5\alpha}} + \Gamma[\tilde{g}]^\alpha_{\alpha\beta}\Gamma[\tilde{g}]^\beta_{55} - \Gamma[\tilde{g}]^\alpha_{5\beta}\Gamma[\tilde{g}]^\beta_{\alpha 5} \\ &= \partial_k\Gamma[\tilde{g}]^k_{55} - \Gamma[\tilde{g}]^5_{5k}\Gamma[\tilde{g}]^k_{55} + \Gamma[\tilde{g}]^\ell_{\ell k}\Gamma[\tilde{g}]^k_{55} \\ &= -Ae^{2A\varphi}\partial_k(g^{kj}\partial_j\varphi) - A^2e^{2A\varphi}g^{kj}\partial_k\varphi\partial_j\varphi - \Gamma[g]^\ell_{\ell k}Ae^{2A\varphi}g^{kj}\partial_j\varphi \\ &= -Ae^{2A\varphi}\Box_g\varphi - A^2e^{2A\varphi}g^{kj}\partial_k\varphi\partial_j\varphi, \end{aligned} \tag{6}$$

where in the last equality we recognized an expression for the Laplace–Beltrami operator. Next, we have

$$\mathrm{Ric}[\tilde{g}]_{ij} = \partial_\alpha\Gamma[\tilde{g}]^\alpha_{ij} - \partial_j\Gamma[\tilde{g}]^\alpha_{i\alpha} + \Gamma[\tilde{g}]^\alpha_{\alpha\beta}\Gamma[\tilde{g}]^\beta_{ij} - \Gamma[\tilde{g}]^\alpha_{i\beta}\Gamma[\tilde{g}]^\beta_{\alpha j}. \tag{7}$$

We now further compute each term on the right side. The terms involving derivatives of Christoffel symbols are

$$\partial_\alpha\Gamma[\tilde{g}]^\alpha_{ij} = \partial_\ell\Gamma[\tilde{g}]^\ell_{ij} = \partial_\ell\Gamma[g]^\ell_{ij}. \tag{8}$$

and

$$\partial_j\Gamma[\tilde{g}]^\alpha_{i\alpha} = \partial_j\Gamma[\tilde{g}]^5_{i5} + \partial_j\Gamma[\tilde{g}]^\ell_{i\ell} = \partial_j\Gamma[g]^\ell_{i\ell} + A\partial_i\partial_j\varphi \tag{9}$$

Next, we compute the terms that are quadratic in the Christoffel symbols. We have

$$\Gamma[\tilde{g}]^\alpha_{\alpha\beta}\Gamma[\tilde{g}]^\beta_{ij} = \Gamma[\tilde{g}]^\ell_{\ell m}\Gamma[\tilde{g}]^m_{ij} + \Gamma[\tilde{g}]^5_{5m}\Gamma[\tilde{g}]^m_{ij} = \Gamma[g]^\ell_{\ell m}\Gamma[g]^m_{ij} + A\Gamma[g]^m_{ij}\partial_m\varphi \tag{10}$$

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and

$$\Gamma[\tilde{g}]^{\alpha}_{i\beta}\Gamma[\tilde{g}]^{\beta}_{\alpha j} = \Gamma[\tilde{g}]^{m}_{i\ell}\Gamma[\tilde{g}]^{\ell}_{mj} + \Gamma[\tilde{g}]^{5}_{i5}\Gamma[\tilde{g}]^{5}_{5j} = \Gamma[g]^{m}_{i\ell}\Gamma[g]^{\ell}_{mj} + A^2\partial_i\varphi\partial_j\varphi. \tag{11}$$

It follows that

$$\mathrm{Ric}[\tilde{g}]_{ij} = \mathrm{Ric}[g]_{ij} - \partial_i\partial_j\varphi - \partial_i\varphi\partial_j\varphi + \Gamma[g]^{m}_{ij}\partial_m\varphi = \mathrm{Ric}[g]_{ij} - A\nabla_i\nabla_j\varphi - A^2\partial_i\varphi\partial_j\varphi. \tag{12}$$

□

We now record the behaviour of various quantities under conformal rescaling.

Lemma 4 (Christoffel symbols under conformal rescaling). *The Christoffel symbols of g are related to those of $h = e^{2B\varphi}g$ by:*

$$\Gamma[g]^{k}_{ij} = \Gamma[h]^{k}_{ij} - B(\delta^k_i\partial_j\varphi + \delta^k_j\partial_i\varphi - h_{ij}h^{k\ell}\partial_\ell\varphi). \tag{13}$$

Lemma 5 (Laplace–Beltrami operator under conformal rescaling). *For $h = e^{2B\varphi}g$, we have*

$$\Box_h f = e^{-2B\varphi}(\Box_g f + B(n-2)g^{ab}\partial_a\varphi\partial_b f). \tag{14}$$

Lemma 6 (Ricci curvature under conformal rescaling). *For $h = e^{2B\varphi}g$, we have*

$$\mathrm{Ric}[h]_{ij} = \mathrm{Ric}[g]_{ij} - B(n-2)\nabla_i\nabla_j\varphi + B^2(n-2)\partial_i\varphi\partial_j\varphi - B(\Box_g\varphi + B(n-2)g^{ab}\partial_a\varphi\partial_b\varphi)g_{ij} \tag{15}$$

where ∇ is the Levi–Civita connection of g.

We can now conclude the proof.

Proof of proposition. If $h = e^{2B\varphi}g$ for $B = A/(n-2)$, then lemma 6 gives

$$\mathrm{Ric}[g]_{ij} - A\nabla_i\nabla_j\varphi = \mathrm{Ric}[h]_{ij} + \frac{A}{n-2}(\Box_g\varphi + Ag^{ab}\partial_a\varphi\partial_b\varphi)g_{ij} - \frac{A^2}{n-2}\partial_i\varphi\partial_j\varphi. \tag{16}$$

It now follows from lemma 3 that

$$\mathrm{Ric}[\tilde{g}]_{ij} = \mathrm{Ric}[h]_{ij} - A^2\Big(1 + \frac{1}{n-2}\Big)\partial_i\varphi\partial_j\varphi + \frac{A}{n-2}h_{ij}e^{-\frac{2}{n-2}A\varphi}\Box_h\varphi, \qquad \mathrm{Ric}[\tilde{g}]_{5j} = 0, \tag{17}$$

$$\mathrm{Ric}[\tilde{g}]_{55} = -Ae^{2A(1+\frac{1}{n-2})\varphi}\Box_h\varphi. \tag{18}$$

The result follows. □

References

[Cic25] Serban Cicortas. "Extensions of lorentzian hawking–page solutions with null singularities, spacelike singularities, and Cauchy horizons of Taub–NUT type". en. *Ann. Henri Poincare* 26.11 (Nov. 2025), pp. 3907–3961.