# MORAWETZ ESTIMATE ON SCHWARZSCHILD

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We follow [HMR24] to derive an integrated local energy decay statement for solutions to the wave equation on the Schwarzschild spacetime (of dimension $(3+1)$).

## 1. Twisted currents

Let $\varphi$ solve $\Box\varphi = F$ and define $\psi \coloneqq \beta^{-1}\varphi$. Define the twisted energy momentum tensor

$$\tilde{T}_{\mu\nu} \coloneqq \beta^2\Big[\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}(\nabla^\alpha\psi\nabla_\alpha\psi + U\psi^2)\Big], \tag{1}$$

where

$$U = -\beta^{-1}\Box\beta. \tag{2}$$

Introduce

$$\tilde{J}^X_\mu \coloneqq \tilde{T}_{\mu\nu}X^\nu \tag{3}$$

and

$$\tilde{K}^X \coloneqq {}^{(X)}\pi^{\mu\nu}\tilde{T}_{\mu\nu} + X^\nu\tilde{S}_\nu \tag{4}$$

and

$$S_\mu \coloneqq -\frac{1}{2}\nabla_\mu(\beta^2 U)\psi^2 - \beta\nabla_\mu\beta\nabla^\nu\psi\nabla_\nu\psi. \tag{5}$$

and

$$\tilde{\mathcal{E}}^X \coloneqq \beta F X\psi. \tag{6}$$

Then

$$\nabla^\mu\tilde{J}^X = \tilde{K}^X + \tilde{\mathcal{E}}^X. \tag{7}$$

Define also

$$\tilde{J}^{\mathrm{aux},w} \coloneqq \beta^2\Big[w\psi\nabla_\mu\psi - \frac{1}{2}\psi^2\nabla_\mu w\Big]. \tag{8}$$

Then

$$\nabla^\mu\tilde{J}^{\mathrm{aux},w}_\mu = \tilde{K}^{\mathrm{aux},w} + \tilde{\mathcal{E}}^{\mathrm{aux},w} \tag{9}$$

for

$$\tilde{K}^{\mathrm{aux},w} \coloneqq \beta^2(wU\psi^2 + w\nabla^\mu\psi\nabla_\mu\psi) - \frac{1}{2}\nabla^\mu(\beta^2\nabla_\mu w)\psi^2 \tag{10}$$

and

$$\tilde{\mathcal{E}}^{\mathrm{aux},w} \coloneqq \beta w\psi\Box\varphi \tag{11}$$

Expanding out the definitions, we have

$$\tilde{K}^X = \beta^2\Big[{}^{(X)}\pi(\mathrm{d}\psi,\mathrm{d}\psi) + \Big(-\frac{1}{2}\operatorname{tr}{}^{(X)}\pi - \beta^{-1}X\beta\Big)|\nabla\psi|^2 + \frac{1}{2}\Big(-\operatorname{tr}{}^{(X)}\pi U - \beta^{-2}X(\beta^2 U)\Big)\psi^2\Big] \tag{12}$$

and

$$\begin{aligned}\tilde{K}^X + \tilde{K}^{\mathrm{aux},w} = \beta^2\Big[&{}^{(X)}\pi(\mathrm{d}\psi,\mathrm{d}\psi) + \Big(w - \frac{1}{2}\operatorname{tr}{}^{(X)}\pi - \beta^{-1}X\beta\Big)|\nabla\psi|^2 \\ &+ \Big(wU - \frac{1}{2}\operatorname{tr}{}^{(X)}\pi U - \frac{1}{2}\beta^{-2}X(\beta^2 U) - \frac{1}{2}\beta^{-2}\nabla^\mu(\beta^2\nabla_\mu w)\Big)\psi^2\Big].\end{aligned} \tag{13}$$

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## 2. Morawetz estimate

The goal of this section is to show the following estimate.

**Proposition 2.1.** *For $M > 0$ we have*

$$\int_{\mathcal{R}(\tau_1,\tau_2)} \Big[\frac{1}{r^2}(\partial_{r^*}\psi)^2 + \frac{1}{r^3}\Big(1-\frac{3M}{r}\Big)^2 |r\nabla\!\!\!/\psi|^2 + \frac{\psi^2}{r^4}\Big] r^{-2}\,\mathrm{dvol} \lesssim_M \int_{\Sigma_{\tau_2}} \tilde{J}^T \cdot n_{\Sigma_{\tau_2}}\,\mathrm{dvol}_{\Sigma_{\tau_2}}. \tag{14}$$

### 2.1. Deformation tensor computations.

**Lemma 2.2.** *Let $Y$ be a vector field and set $X = f(r)Y$. Then*

$${}^{(X)}\pi_{\alpha\beta} = f\,{}^{(Y)}\pi_{\alpha\beta} + \frac{1}{2}(\nabla_\alpha f Y_\beta + \nabla_\beta f Y_\alpha). \tag{15}$$

*and*

$$\mathrm{tr}\,{}^{(X)}\pi = f\,\mathrm{tr}\,{}^{(Y)}\pi + Yf. \tag{16}$$

*If $Y$ is a coordinate vector field, then*

$${}^{(Y)}\pi_{\alpha\beta} = \frac{1}{2}Yg_{\alpha\beta}. \tag{17}$$

*Proof.* Using $X = fY$, compute

$${}^{(X)}\pi_{\alpha\beta} = f\,{}^{(Y)}\pi_{\alpha\beta} + \frac{1}{2}(\nabla_\alpha f Y_\beta + \nabla_\beta f Y_\alpha). \tag{18}$$

From (15) we obtain

$$\mathrm{tr}\,{}^{(X)}\pi = f\,\mathrm{tr}\,{}^{(Y)}\pi + Yf. \tag{19}$$

Using the definition ${}^{(Y)}\pi = \frac{1}{2}\mathcal{L}_Y g$, we compute (assuming $Y$ is a coordinate vector field so that $\mathcal{L}_Y\partial_\alpha = 0$)

$$2\,{}^{(Y)}\pi_{\alpha\beta} = (\mathcal{L}_Y g)(\partial_\alpha,\partial_\beta) = Yg(\partial_\alpha,\partial_\beta) - g(\mathcal{L}_Y\partial_\alpha,\partial_\beta) - g(\partial_\alpha,\mathcal{L}_Y\partial_\beta) = Yg_{\alpha\beta} \tag{20}$$

□

**Lemma 2.3.** *Write $Y = \partial_{r^*}$ and set $X = f(r)\partial_{r^*} = f(r)Y$. Write $f' = \partial_{r^*}f$. We have*

$$\mathrm{tr}\,{}^{(X)}\pi = \Big(\frac{2}{r} - \frac{2M}{r^2}\Big)f + f'. \tag{21}$$

*and*

$${}^{(X)}\pi(\mathrm{d}\varphi,\mathrm{d}\varphi) = \frac{f'}{1-\mu}(\partial_{r^*}\varphi)^2 + \frac{f}{r}\Big(1-\frac{3M}{r}\Big)|\nabla\!\!\!/\varphi|^2 + \frac{fM}{r^2}|\nabla\varphi|^2. \tag{22}$$

*Proof.* Using (17) and the form of the Schwarzschild metric, compute that the non-zero components of ${}^{(Y)}\pi$ are

$$-{}^{(Y)}\pi_{tt} = {}^{(Y)}\pi_{r^*r^*} = \frac{M}{r^2}(1-\mu) \qquad {}^{(Y)}\pi_{ab} = \frac{1}{r}(1-\mu)g\!\!\!/_{ab}. \tag{23}$$

It follows immediately that

$$\mathrm{tr}\,{}^{(Y)}\pi = \frac{2M}{r^2} + \frac{2}{r}(1-\mu) = \frac{2}{r} - \frac{2M}{r^2} \tag{24}$$

and hence

$$\mathrm{tr}\,{}^{(X)}\pi = \Big(\frac{2}{r} - \frac{2M}{r^2}\Big)f + f'. \tag{25}$$

Since the only non-zero component of $Y$ with down indices is $Y_{r^*} = g_{r^*r^*}Y^{r^*} = g_{r^*r^*} = (1-\mu)$, and $f$ is a function only of $r$, the second term in (15) contributes only when $\alpha = \beta = r^*$. It follows that

$${}^{(X)}\pi_{tt} = -\frac{fM}{r^2}(1-\mu) \quad {}^{(X)}\pi_{r^*r^*} = \Big[\frac{fM}{r^2} + f'\Big](1-\mu) \quad {}^{(X)}\pi_{ab} = \frac{f}{r}(1-\mu)g\!\!\!/_{ab}. \tag{26}$$

We now compute

$$\begin{aligned}{}^{(X)}\pi(\mathrm{d}\varphi,\mathrm{d}\varphi) = {}^{(X)}\pi^{\alpha\beta}\partial_\alpha\partial_\beta\varphi = {}^{(X)}\pi^\alpha_\beta\partial^\alpha\varphi\partial_\beta\varphi &= \frac{fM}{r^2}\partial^t\varphi\partial_t\varphi + \Big[\frac{fM}{r^2} + f'\Big]\partial^{r^*}\varphi\partial_{r^*}\varphi + \frac{f}{r}(1-\mu)|\nabla\!\!\!/\varphi|^2 \\ &= f'\partial^{r^*}\varphi\partial_{r^*}\varphi + \Big[\frac{f}{r}(1-\mu) - \frac{fM}{r^2}\Big]|\nabla\!\!\!/\varphi|^2 + \frac{fM}{r^2}|\nabla\varphi|^2 \\ &= \frac{f'}{1-\mu}(\partial_{r^*}\varphi)^2 + \frac{f}{r}\Big(1-\frac{3M}{r}\Big)|\nabla\!\!\!/\varphi|^2 + \frac{fM}{r^2}|\nabla\varphi|^2.\end{aligned} \tag{27}$$

$\square$

2.2. **Computing $\tilde{K}^f$.** Define

$$\tilde{K}^f[\psi] \coloneqq \tilde{K}^{f(r)\partial_{r^*}}[\psi] + \tilde{K}^{\mathrm{aux},f'/2}[\psi]. \tag{28}$$

**Lemma 2.4.** *We have*

$$\tilde{K}^f[\psi] = r^{-2}\frac{1}{1-\mu}\Big[f'(\partial_{r^*}\psi)^2 + \frac{f}{r^3}(1-\mu)\Big(1-\frac{3M}{r}\Big)|\nabla_{S^2}\psi|^2 - \Big(\frac{1}{2}f[U(1-\mu)]' + \frac{1}{4}f'''\Big)\psi^2\Big]. \tag{29}$$

*Proof.* We have

$$\begin{aligned}\tilde{K}^X + \tilde{K}^{\mathrm{aux},w} = r^{-2}\Big[&\frac{f'}{1-\mu}(\partial_{r^*}\psi)^2 + \frac{f}{r^3}\Big(1-\frac{3M}{r}\Big)|\nabla_{S^2}\psi|^2 + \Big(w - \frac{f'}{2}\Big)|\nabla\psi|^2 \\ &+ \Big(wU - \frac{1}{2}\operatorname{tr}{}^{(X)}\pi U - \frac{1}{2}\beta^{-2}X(\beta^2 U) - \frac{1}{2}\beta^{-2}\nabla^\mu(\beta^2\nabla_\mu w)\Big)\psi^2\Big]\end{aligned} \tag{30}$$

Note that $|\nabla\!\!\!/\psi| = r^{-1}|\nabla_{S^2}\psi|$. Take $w = f'/2$ to make the $|\nabla\psi|$ term vanish, and compute

$$wU = \frac{f'}{2}U \tag{31}$$

and

$$-\frac{1}{2}\operatorname{tr}{}^{(X)}\pi U = -\Big(1-\frac{M}{r}\Big)\frac{f}{r}U - \frac{f'}{2}U. \tag{32}$$

and

$$-\frac{1}{2}\beta^{-2}X(\beta^2 U) = -\frac{1}{2}XU - \beta^{-1}X\beta U = -\frac{1}{2}fU' + \frac{f}{r}(1-\mu)U \tag{33}$$

and

$$\Box w = \frac{1}{(1-\mu)r^2}\partial_{r^*}(r^2\partial_{r^*}w) = \frac{1}{1-\mu}\partial^2_{r^*}w + \frac{2}{r}\partial_{r^*}w = \frac{1}{2}\frac{1}{1-\mu}f''' + \frac{1}{r}f'', \tag{34}$$

and

$$-\beta^{-1}\nabla^\mu\beta\nabla_\mu w = -r\nabla^{r^*}r^{-1}\nabla_{r^*}w = \frac{1}{2}\frac{1}{r}f'', \tag{35}$$

so that

$$-\frac{1}{2}\beta^{-2}\nabla^\mu(\beta^2\nabla_\mu w) = -\beta^{-1}\nabla^\mu\beta\nabla_\mu w - \frac{1}{2}\Box w = -\frac{1}{4}\frac{1}{1-\mu}f'''. \tag{36}$$

$\square$

2.3. **Obtaining the reduced estimate.** Split $\psi$ into $\psi_0$ and $\psi_{\geq 1} \coloneqq \psi - \psi_0$, where $\psi_0$ is the spherical mean of $\psi$ (the zeroth spherical mode). By the orthogonality of $\psi_{\geq 1}$ and $\psi_0$ on $S^2$, we have

$$\int_{S^2}\tilde{K}^f[\psi]\,\mathrm{d}\sigma = \int_{S^2}\tilde{K}^f[\psi_{\geq 1}] + \tilde{K}^f[\psi_0]\,\mathrm{d}\sigma. \tag{37}$$

2.3.1. *Estimates for $\tilde{K}^f[\psi_{\geq 1}]$.* The smallest non-zero eigenvalue of $-\Delta_{S^2}$ is 2, so

$$\int_{S^2}\tilde{K}^f[\psi_{\geq 1}]\,\mathrm{d}\sigma \geq r^{-2}\frac{1}{1-\mu}\Big[f'(\partial_{r^*}\psi_{\geq 1})^2 + \Big(f\Big[\frac{2}{r^3}(1-\mu)\Big(1-\frac{3M}{r}\Big) - \frac{1}{2}[U(1-\mu)]'\Big] - \frac{1}{4}f'''\Big)\psi_{\geq 1}^2\Big]. \tag{38}$$

It follows from the next two lemmas that

$$\int_{\mathcal{R}(\tau_1,\tau_2)}\tilde{K}^f[\psi_{\geq 1}]\,\mathrm{dvol} \geq c(M)\int_{\mathcal{R}(\tau_1,\tau_2)}\Big[\frac{1}{r^3}(\partial_{r^*}\psi_{\geq 1})^2 + \frac{1}{r^3}|\psi_{\geq 1}|^2\Big]r^{-2}\,\mathrm{dvol}. \tag{39}$$

**Lemma 2.5.** *For*

$$f = \Big(1-\frac{3M}{r}\Big)\Big(1+\frac{M}{r}\Big), \tag{40}$$

*we have*

$$f\Big[\frac{2}{r^3}(1-\mu)\Big(1-\frac{3M}{r}\Big) - \frac{1}{2}[U(1-\mu)]'\Big] - \frac{1}{4}f''' = \frac{2}{r^7}\Big(1-\frac{2M}{r}\Big)P(r) \tag{41}$$

*for*

$$P(r) = r^4 - 5Mr^3 - 3M^2r^2 + 50M^3r - 60M^4 \tag{42}$$

*Proof.* Here are the intermediate steps in this computation. We have

$$f' = \frac{2M}{r^2}\Big(1 - \frac{2M}{r}\Big)\Big(1 + \frac{3M}{r}\Big) \tag{43}$$

and

$$f'' = \frac{2M}{r^5}\Big(1 - \frac{2M}{r}\Big)(24M^2 - 3Mr - 2r^2) \tag{44}$$

and

$$f''' = \frac{4M}{r^7}\Big(1 - \frac{2M}{r}\Big)(144M^3 - 75M^2r - 2Mr^2 + 3r^3). \tag{45}$$

Compute

$$f\frac{2}{r^3}(1-\mu)\Big(1 - \frac{3M}{r}\Big) = \frac{1}{r^7}\Big(1 - \frac{2M}{r}\Big)(2r^4 - 10Mr^3 + 6M^2r^2 + 18M^3r) \tag{46}$$

and

$$-\frac{1}{2}f[U(1-\mu)]' = \frac{1}{r^7}\Big(1 - \frac{2M}{r}\Big)(3Mr^3 - 14M^2r^2 + 7M^3r + 24M^4) \tag{47}$$

and

$$-\frac{1}{4}f''' = \frac{1}{r^7}\Big(1 - \frac{2M}{r}\Big)(-3Mr^3 + 2M^2r^2 + 75M^3r - 144M^4). \tag{48}$$

Add the previous three equations to conclude. □

**Lemma 2.6.** *For $r \geq 2M$ we have $P(x) > 0$.*

*Proof.* By scaling we need to show that

$$\tilde{P}(r) = r^4 - 5r^3 - 3r^2 + 50r - 60 \tag{49}$$

is positive for $r \geq 2$. Since $\tilde{P}(2) = 4 \geq 0$, it is enough to show that

$$\tilde{P}'(r) = 4r^3 - 15r^2 - 6r + 50 \tag{50}$$

is positive for $r \geq 2$. Note that $\tilde{P}'(2) = 10$ and $\tilde{P}'$ is positive at the largest root of

$$\tilde{P}''(r) = 12r^2 - 30r - 6. \tag{51}$$

This shows that $\tilde{P}'(r) > 0$ in $r \geq 2$, as desired. □

2.3.2. *Estimates for $\tilde{K}^f[\psi_0]$.* Let $g = g(r)$ be a function. Note that

$$\tilde{K}^f[\psi_0] = r^{-2}\frac{1}{1-\mu}\Big[f'|\psi_0'|^2 - \Big(\frac{f}{2}[U(1-\mu)]' + \frac{1}{4}f'''\Big)|\psi_0|^2\Big]. \tag{52}$$

with

$$\begin{aligned} &f'|\psi_0'|^2 - \Big(\frac{f}{2}[U(1-\mu)]' + \frac{1}{4}f'''\Big)|\psi_0|^2 \\ &= f'|\psi_0' - g\psi_0|^2 + 2f'g\psi_0'\psi_0 - \Big(f'g^2 + \frac{f}{2}[U(1-\mu)]' + \frac{1}{4}f'''\Big)|\psi_0|^2 \\ &= f'|\psi_0' - g\psi_0|^2 + (f'g|\psi_0|^2)' - \Big(f'g^2 + (f'g)' + \frac{f}{2}[U(1-\mu)]' + \frac{1}{4}f'''\Big)|\psi_0|^2. \end{aligned} \tag{53}$$

**Lemma 2.7.** *There is a function $g(r)$ such that*

$$-\Big(f'g^2 + (f'g)' + \frac{f}{2}[U(1-\mu)]' + \frac{1}{4}f'''\Big) > \frac{c(M)}{r^4}\Big(1 - \frac{2M}{r}\Big). \tag{54}$$

*Proof.* With

$$g(r) = \frac{1}{2r}\Big(1 - \frac{2M}{r} - \frac{M^2}{r^2}\Big), \tag{55}$$

we compute

$$f'g^2 = \frac{1}{2r^9}\Big(1 - \frac{2M}{r}\Big)(3M^6 + 13M^5r + 10M^4r^2 - 10M^3r^3 - M^2r^4 + Mr^5). \tag{56}$$

and

$$(f'g)' = \frac{1}{2r^9}\Big(1 - \frac{2M}{r}\Big)(-84M^5r - 132M^4r^2 + 90M^3r^3 + 8M^2r^4 - 6Mr^5) \tag{57}$$

and
$$\frac{f}{2}[U(1-\mu)]' = \frac{1}{2r^9}\Big(1-\frac{2M}{r}\Big)(-48M^4r^2 - 14M^3r^3 + 28M^2r^4 - 6Mr^5) \tag{58}$$
and
$$\frac{1}{4}f''' = \frac{1}{2r^9}\Big(1-\frac{2M}{r}\Big)(288M^4r^2 - 150M^3r^3 - 4M^2r^4 + 6Mr^5). \tag{59}$$
The negative of the sum of the previous four equations is
$$-\Big(f'g^2 + (f'g)' + \frac{f}{2}[U(1-\mu)]' + \frac{1}{4}f'''\Big) = \frac{1}{2r^9}\Big(1-\frac{2M}{r}\Big)P(x) \tag{60}$$
for
$$P(r) = 5Mr^5 - 31M^2r^4 + 84M^3r^3 - 118M^4r^2 + 71M^5r - 3M^6. \tag{61}$$
It is now enough to show that $P(r) > 0$ for $r \geq 2M$. By scaling we can suppose $M = 1$. One easily checks that $P^{(i)}(2M)$ is positive for all $0 \leq i \leq 4$. □

It follows that
$$\tilde{K}^f[\psi_0] \geq r^{-2}\frac{1}{1-\mu}\Big[(f'g|\psi_0|^2)' + \frac{c(M)}{r^4}(1-\mu)|\psi_0|^2\Big]. \tag{62}$$

2.3.3. *Boundary terms.* We have
$$\tilde{J}^T \cdot T = \beta^2\Big[\frac{1}{2}(T\psi)^2 + \frac{1}{2}(\partial_{r^*}\psi)^2 + \frac{1}{2}|\nabla\!\!\!/\psi|^2 + U\psi^2\Big], \tag{63}$$
and
$$\tilde{J}^T \cdot L = \beta^2\Big[\frac{1}{2}(L\psi)^2 + \frac{1}{2}(1-\mu)(|\nabla\!\!\!/\psi|^2 + U\psi^2)\Big], \tag{64}$$
and
$$\tilde{J}^T \cdot \underline{L} = \beta^2\Big[\frac{1}{2}(\underline{L}\psi)^2 + \frac{1}{2}(1-\mu)(|\nabla\!\!\!/\psi|^2 + U\psi^2)\Big], \tag{65}$$
Similarly, we compute
$$\tilde{J}^X \cdot T = |\beta^2 fX\psi T\psi| \lesssim \tilde{J}^T \cdot T \tag{66}$$
and
$$|\tilde{J}^X \cdot L| = \beta^2 f\Big|\frac{1}{2}(L\psi)^2 - \frac{1}{2}(1-\mu)(|\nabla\!\!\!/\psi|^2 + U\psi^2)\Big| \lesssim \tilde{J}^T \cdot L \tag{67}$$
and
$$|\tilde{J}^X \cdot \underline{L}| = \beta^2 f\Big|\frac{1}{2}(\underline{L}\psi)^2 + \frac{1}{2}(1-\mu)(|\nabla\!\!\!/\psi|^2 + U\psi^2)\Big| \lesssim \tilde{J}^T \cdot \underline{L}. \tag{68}$$
It follows that the boundary terms arising from $\tilde{J}^X$ can be controlled by those arising from $\tilde{J}^T$. Moreover, we have
$$\int_{\mathcal{R}(\tau_1,\tau_2)} \partial_r(f'g|\psi_0|^2)r^{-2}\,\mathrm{dvol} = \int_{\mathcal{R}(\tau_1,\tau_2)} \partial_r(f'g|\psi_0|^2)\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\sigma, \tag{69}$$
which contributes boundary terms of size $f'g|\psi_0|^2 \lesssim r^{-3}(1-\mu)|\psi_0|^2$, which can evidently be controlled by the zeroth order term in $\tilde{J}^T$.

2.3.4. *Completing the estimate.* We conclude that
$$\begin{aligned}
&\int_{\mathcal{R}(\tau_1,\tau_2)} \Big[\frac{1}{r^2}(\partial_{r^*}\psi)^2 + \frac{1}{r^3}\Big(1-\frac{3M}{r}\Big)^2|r\nabla\!\!\!/\psi|^2 + \frac{\psi^2}{r^4}\Big]r^{-2}\,\mathrm{dvol} \\
&\lesssim \int_{\mathcal{R}(\tau_1,\tau_2)} \tilde{K}^f[\psi_0]\,\mathrm{dvol} + \Big|\int_{\mathcal{R}(\tau_1,\tau_2)} \partial_r(f'g|\psi_0|^2)r^{-2}\,\mathrm{dvol}\Big| \lesssim \int_{\partial\mathcal{R}(\tau_1,\tau_2)} \tilde{J}^T \cdot n\,\mathrm{dvol}_{\partial\mathcal{R}(\tau_1,\tau_2)} \\
&\lesssim \int_{\Sigma_{\tau_2}} \tilde{J}^T \cdot n_{\Sigma_{\tau_2}}\,\mathrm{dvol}_{\Sigma_{\tau_2}},
\end{aligned} \tag{70}$$
where in the last line we have used the energy identity associated to $\tilde{J}^T$.

## References

[HMR24] Gustav Holzegel, Georgios Mavrogiannis, and Renato Velozo Ruiz. *A note on integrated local energy decay estimates for spherically symmetric black hole spacetimes.* 2024. arXiv: 2403.02533 [gr-qc].