THE WEYL TUBE FORMULA

ONYX GAUTAM

These are notes for the Princeton Graduate Student Seminar talk given on November 20, 2025. References include [Wey39; Hot39; Gra03].

1. Introduction

Let $\Gamma \subset \mathbf{R}^2$ be an embedded plane curve. Let $\mathcal{T}_\epsilon$ be the *tube* around Γ of width ϵ (in both directions). That is, let $\mathcal{T}_\epsilon$ be the set of points from which there is a straight line to Γ of length at most ϵ that meets Γ orthogonally.

Remark 1.1. When Γ is smooth and closed and ϵ is sufficiently small, $\mathcal{T}_\epsilon$ is the set of points within distance ϵ of Γ. For a non-closed curve, this latter set is larger than $\mathcal{T}_\epsilon$, due to the presence of "end caps."

What is the area of $\mathcal{T}_\epsilon$?

Example 1.2 (Line segment). A line segment of length L produces a tube (rectangle) of area $2L\epsilon$.

Example 1.3 (Circle). A circle of radius r produces a tube of area

$$\pi(r+\epsilon)^2 - \pi(r-\epsilon)^2 = 4\pi\epsilon = 2 \cdot 2\pi \cdot \epsilon \tag{1}$$

Example 1.4 (Square). Note that a square is not C^2. A square of side length r produces a tube of area

$$2 \cdot 4r\epsilon - 4\epsilon^2, \tag{2}$$

since the measurement $8r\epsilon$ double counts each corner.

Theorem 1.5 (Weyl tube formula for plane curves). *If Γ is C^2, then when ϵ is sufficiently small, we have*

$$\mathrm{Vol}(\mathcal{T}_\epsilon) = 2L\epsilon. \tag{3}$$

Remark 1.6. This is an exact equality! There is no $O(\epsilon^2)$ term.

Remark 1.7 (Smallness of ϵ). The tube width ϵ needs to be small enough that the tubular neighbourhood does not overlap itself (each point in the tube has a unique closest point on the curve). For this we can take ϵ to be smaller than the inverse of the maximum curvature of the curve.

Remark 1.8 (Historical origin of the problem). In a 1938 lecture before the Princeton Mathematics Club, the mathematical statistician and economist Harold Hotelling posed the following problem:

> Consider a closed submanifold of $\mathbf{R}^n$, and consider the set of points of distance at most ϵ from the submanifold. What is the volume of this region?

The study of this problem goes back to Steiner in 1840 [Ste13], who considered the case of piecewise linear closed convex curves in $\mathbf{R}^2$ and convex surfaces in $\mathbf{R}^3$. The problem was solved for curves in $\mathbf{R}^n$ by Hotelling [Hot39] and in full generality by Weyl [Wey39].

Remark 1.9 (Hotelling's law). Hotelling is well known for Hotelling's law [Hot29]. This is the observation that in many economic markets, it is rational for producers to make their products as similar as possible. For example, if there are two shops on a street selling the same products at the same prices, then both shops will be next to each other, at the halfway point of the street.

Remark 1.10 (Hotelling's PhD). After graduating from the University of Washington with a degree in journalism, Hotelling came to Princeton to study mathematical economics with Thorstein Veblen. Unfortunately, it was not Thorstein Veblen who worked at Princeton, but his nephew Oswald Veblen. It wasn't until he actually arrived at Princeton that he realized his mistake, but he made the best of it and did a PhD in

Date: November 20, 2025.

topology (or "analysis situs," as it was called at the time) on three-dimensional manifolds. The thesis was called "Three-dimensional manifolds of states of motion." Apparently Hotelling related this story to Albert Tucker, who gave an interview in 1984 [Asp85], 12 years after Hotelling's death.

2. Proof of the Weyl tube formula for plane curves

Suppose Γ is smooth (though it will be clear that C^2 is enough). Orient Γ by choosing a normal vector field, which by convention we take to be inward-pointing. Let Γ_t be the set of points from which there is a line segment of signed length t to Γ meeting Γ orthogonally, where the sign is determined by the direction of the normal vector field. For small enough t, the set Γ_t is a smooth curve, and

$$\mathrm{Vol}(\mathcal{T}_\epsilon) = \int_{-\epsilon}^{\epsilon} \mathrm{Len}(\Gamma_t)\,\mathrm{d}t. \tag{4}$$

Let $\gamma : [a,b] \to \mathbf{R}^2$ be a smooth parametrization of the curve Γ. We suppose that γ is a unit speed parametrization, namely that

$$|\gamma'(s)| = 1 \text{ for all } s \in [a,b]. \tag{5}$$

This can always be achieved by a reparametrization. Then we say that γ parameterized by arc-length, and

$$\mathrm{Len}(\Gamma) = \int_a^b |\gamma'(s)|\,\mathrm{d}s = b-a. \tag{6}$$

Definition 2.1 (Velocity and acceleration)**.** We call $\gamma'(s)$ the *velocity*. We call $\gamma''(s)$ the acceleration.

Remark 2.2 (Properties of velocity and acceleration). Note that $\gamma'(s)$ is non-zero and tangent to the curve. Moreover, $\gamma''(s)$ is perpendicular to $\gamma'(s)$, as can be seen by differentiating the formula $\gamma' \cdot \gamma' = 1$. This only holds because γ is unit speed. For a general parametrization, part of the acceleration vector will measure the deviation from unit speed.

Now consider the map

$$J(x,y) = (-y,x). \tag{7}$$

This is counterclockwise rotation by 90 degrees; in the complex plane, this is multiplication by i.

Definition 2.3 (Curvature)**.** Since $\gamma'(s)$ is non-zero, there exists a unique function $\kappa : [a,b] \to \mathbf{R}$ such that

$$\gamma''(s) = \kappa(s) \cdot J\gamma'(s). \tag{8}$$

We call κ the *(signed) curvature*.

Remark 2.4 (The curvature (a scalar) and acceleration (a vector) have the same magnitude). Note that $|\kappa(s)| = |\gamma''(s)|$. Moreover, the curvature is a quantity associated to the curve, since an oriented curve has exactly one arc-length parametrization.

If we introduce the notation $\mathbf{T}(s) \coloneqq \gamma'(s)$ and $\mathbf{N}(s) \coloneqq J\gamma'(s)$, then we have the *Frenet–Serret equations*:

$$\begin{cases} \mathbf{T}'(s) & = \kappa(s)\mathbf{N}(s) \\ \mathbf{N}'(s) & = -\kappa(s)\mathbf{T}(s), \end{cases} \tag{9}$$

where we used the identity $J^2 = -1$. Now define

$$\gamma_t(s) = \gamma(s) + t\mathbf{N}(s). \tag{10}$$

Then $\gamma_0(s) = \gamma(s)$, and for small fixed t, the map $\gamma_t : [a,b] \to \mathbf{R}^2$ traces out a curve parallel to γ. That is, γ_t is a parametrization of Γ_t (which is not necessarily unit-speed). To see this, note that from (9), it follows that

$$\gamma_t'(s) = \gamma'(s) - \kappa(s)\mathbf{T}(s) = (1-\kappa(s)t)\mathbf{T}(s). \tag{11}$$

Remark 2.5. In particular, if $|t| < (\max_\Gamma \kappa)^{-1}$, then $\gamma_t' \neq 0$, and so $s \mapsto \gamma_t(s)$ is a smooth curve.

In particular, we have

$$\mathrm{Len}(\Gamma_t) = \int_a^b |\gamma_t'(s)|\,\mathrm{d}s = \int_a^b (1-\kappa(s)t)\,\mathrm{d}s = \mathrm{Len}(\Gamma) - t\int_\Gamma \kappa\,\mathrm{d}s. \tag{12}$$

Integrating this over $t \in [-\epsilon, \epsilon]$, we get the desired formula:

$$\mathrm{Vol}(\mathcal{T}_\epsilon) = \int_{-\epsilon}^{\epsilon} \mathrm{Len}(\Gamma_t)\,\mathrm{d}t = 2\epsilon \cdot \mathrm{Len}(\Gamma) - \cancel{\int_{-\epsilon}^{\epsilon} t\Big(\int_\Gamma \kappa\,\mathrm{d}s\Big)}\,\mathrm{d}t. \tag{13}$$

Remark 2.6 (Total curvature)*.* The quantity $\int_\Gamma \kappa\,\mathrm{d}s$ is called the *total curvature* of Γ. A *closed* embedded plane curve always has total curvature 2π. A closed *immersed* curve has total curvature $2\pi n$, where n is the *turning number*, namely the winding number of the tangent vector around the origin (equivalently, the degree of the map $S^1 \to S^1$ taking a point on the curve to its unit tangent vector).

3. The Weyl tube formula for surfaces in $\mathbf{R}^3$

Theorem 3.1 (Weyl tube formula for plane curves)**.** *Consider a closed surface $\Sigma \subset \mathbf{R}^3$. Let $\mathcal{T}_\epsilon$ be the tubular neighbourhood of Σ (in both directions) of width ϵ. Then*

$$\mathrm{Vol}(\mathcal{T}_\epsilon) = 2\mathrm{Area}(\Sigma)\epsilon + \frac{4\pi}{3}\chi(\Sigma)\epsilon^3, \tag{14}$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface.

Remark 3.2*.* This is independent of the embedding of the surface!

Remark 3.3*.* The coefficients 2 and $4\pi/3$ are the volumes of the unit 1-ball and unit 3-ball, respectively.

3.1. **Differential geometry of surfaces in $\mathbf{R}^3$.** Let $\Sigma \subset \mathbf{R}^3$ be a smooth oriented surface with unit normal vector field n.

Definition 3.4 (Shape operator)**.** The shape operator $S : T\Sigma \to T\Sigma \subset T\mathbf{R}^3$ is defined by

$$SX = -\nabla_X n = -\nabla_X(n^i\partial_i) = -(Xn^i)\partial_i \tag{15}$$

where n is a unit normal vector field to Σ.

Remark 3.5*.* To see that $SX \in T\Sigma$, differentiate $n \cdot n = 1$ in the X-direction to get $\nabla_X n \cdot n = 0$.

Remark 3.6*.* The shape operator of X captures how the tangent planes of Σ are changing in the X direction.

Example 3.7 (Shape operator of a plane)*.* The shape operator of a plane vanishes, since the normal vector is constant.

Example 3.8 (Shape operator of a sphere)*.* Let $\Sigma = \{x \in \mathbf{R}^3 : |x| = r\}$ be a sphere of radius r. Let n be the outward normal on Σ. We have

$$\nabla_X n|_{(x^1,\ldots,x^3)} = -X\Big(\frac{x^i}{r}\Big)\partial_i = -\frac{1}{r}(Xx^i)\partial_i = -\frac{1}{r}X^i\partial_i = -\frac{1}{r}X. \tag{16}$$

That is, the shape operator is multiplication by the constant $-1/r$.

Example 3.9 (Shape operator of a cylinder)*.* Consider a cylinder of radius r. The shape operator in the "flat" direction of the cylinder vanishes, but the shape operator in the "curved" direction is multiplication by $-1/r$.

Definition 3.10 (Mean curvature and Gaussian curvature)**.** We call $H \coloneqq \operatorname{tr} S$ the *mean curvature* and $K \coloneqq \det S$ the *Gaussian curvature*.

The Gaussian curvature is famously *intrinsic* to the surface.

Theorem 3.11 (Theorema Egregium)**.** *The Gaussian curvature of a surface is invariant under isometry.*

Moreover, the integral of the Gaussian curvature, a geometric quantity, is related to the Euler characteristic, a topological quantity.

Theorem 3.12 (Gauss–Bonnet)**.** *Let Σ be a closed Riemannian surface, and let K be the Gaussian curvature of Σ. Then*

$$\int_\Sigma K\,\mathrm{d}A = 2\pi\chi(\Sigma), \tag{17}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

3.2. **Proof of the Weyl tube formula for surfaces in $\mathbf{R}^3$.**

Proof. Choose a local parametrization $\sigma : \mathcal{U} \to \Sigma \subset \mathbf{R}^3$ of Σ. Let N be a normal vector field for Σ, and let $n : \mathcal{U} \to \Sigma$ assign to a point $(u, v) \in \mathcal{U}$ the normal vector to σ at $\sigma(u, v)$, namely $N|_{\sigma(u,v)}$. Let $S : X \mapsto -\nabla_X n$ be the shape operator of σ. Note that the partial derivatives σ_u and σ_v are tangent vector fields, and

$$S\sigma_u = -n_u, \qquad S\sigma_v = -n_v. \tag{18}$$

Now define

$$\sigma^t(u, v) = \sigma(u, v) + tn(u, v). \tag{19}$$

For small enough t, this map is injective and parametrizes a surface Σ^t at signed distance t to Σ. We have

$$\sigma^t_u = \sigma_u + tn_u = (1_{T\Sigma} - tS)\sigma_u, \qquad \sigma^t_v = (1_{T\Sigma} - tS)\sigma_v. \tag{20}$$

Recalling that the magnitude of the cross product of two vectors is the area of the parallelogram they span, we compute

$$\begin{aligned}
&\mathrm{Area}(\sigma^t(U)) \\
&= \int_{\sigma(U)} \mathrm{d}A \\
&= \int_U \|\sigma^t_u \times \sigma^t_v\|\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_U \|((1 - tS)\sigma_u) \times ((1 - tS)\sigma_v)\|\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_U |\det(1 - tS)|\|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_U \det(1 - tS)\|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v \qquad \text{(for } t \text{ sufficiently small)} \\
&= \int_U (1 - t\,\mathrm{tr}\,S + t^2\det S)\|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v \qquad \text{since } \det\begin{bmatrix} 1-a & -b \\ -c & 1-d \end{bmatrix} = 1 - (a + d) + (ad - bc) \\
&= \mathrm{Area}(\sigma(U)) - t\int_{\sigma(U)} \mathrm{tr}\,S\,\mathrm{d}A + t^2\int_{\sigma(U)} \det S\,\mathrm{d}A.
\end{aligned} \tag{21}$$

Piecing together the local parametrizations, we get

$$\mathrm{Area}(\Sigma^t) = \mathrm{Area}(\Sigma) - t\int_\Sigma H\,\mathrm{d}A + t^2\int_\Sigma K\,\mathrm{d}A. \tag{22}$$

Integrating over $t \in [-\epsilon, \epsilon]$, we get

$$\mathrm{Vol}(\mathcal{T}_\epsilon) = \int_{-\epsilon}^{\epsilon} \mathrm{Area}(\Sigma^t)\,\mathrm{d}t = 2\epsilon\mathrm{Area}(\Sigma) + \frac{2\epsilon^3}{3}\int_\Sigma K\,\mathrm{d}A = 2\epsilon\mathrm{Area}(\Sigma) + \frac{4\pi}{3}\epsilon^3, \tag{23}$$

where in the final equality we used the Gauss–Bonnet theorem. □

4. The Weyl tube formula in arbitrary dimension and codimension

Theorem 4.1 (Weyl tube formula)**.** *Let $S \subset \mathbf{R}^n$ be a closed submanifold of dimension m. The tube $\mathcal{T}_\epsilon$ of width ϵ around S has volume*

$$\mathrm{Vol}(\mathcal{T}_\epsilon) = \sum_{j=0}^{\lfloor m/2 \rfloor} C_{n,m,j}k_{2j}(S)\epsilon^{2j+n-m}, \tag{24}$$

where $C_{n,m,j}$ are explicit constants involving factorials and the volume of the unit ball in various dimensions, and $k_{2j}(S)$ are integrals of curvature invariants.

Remark 4.2. This is a polynomial in ϵ of degree $2\lfloor m/2\rfloor + \operatorname{codim} S$. When S is a plane curve, this is 1, and when S is a surface in $\mathbf{R}^3$, this is 3.

Remark 4.3. The volume is independent of the embedding of S! It depends only on the induced metric of the submanifold. The coefficients $k_{2j}(S)$ have natural expressions in terms of the second fundamental form (or shape operator) of the surface, which *does* depend on the embedding. Indeed, after computing these expressions Weyl says:

> So far we have hardly done more than what could have been accomplished by any student in a course of calculus. However, some less obvious argument is needed for [the rest of the proof].

The deep part of Weyl's theorem is that these coefficients can actually be expressed in terms of curvature invariants. The proof uses a combination of the Gauss equation and Weyl's theory of invariants.

Remark 4.4 (Examples of curvature invariants appearing in the Weyl tube formula). For example, we have

$$k_2(S) = \frac{1}{2}\int_S R\,\mathrm{d}S, \tag{25}$$

where R is the scalar curvature of S. The scalar curvature of a surface is simply twice the Gaussian curvature. When $\dim S = m$ is even, we have

$$k_m(S) = (2\pi)^{m/2}\chi(S). \tag{26}$$

Remark 4.5 (Riemann curvature or "vector vortex"?). In [Wey39], Weyl says:

> It is a pity that the inadequate name "curvature," which ought to be reserved for [the shape operator], has been attached to the Riemann tensor. In [a previous paper], I proposed the more descriptive term "vector vortex" [(*in German "Vektorwirbel"*)].

The origin of Weyl's proposed terminology is probably the following: parallel transporting a vector around a loop in Euclidean space returns the same vector. This is not the case in a general Riemannian manifold. Indeed, if a vector Z is parallel transported around a quadrilateral with sides given by tY, sX, $-tY$, and $-sX$, then as $t, s \to 0$, the vector obtained is

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z = R(X,Y)Z, \tag{27}$$

which is the Riemann curvature tensor.

Remark 4.6 (Applications and generalizations of Gauss–Bonnet, following [Gra03, Sec. 5.6]). In 1926, Hopf [Hop26] generalized the Gauss–Bonnet theorem to closed hypersurfaces in odd-dimensional Euclidean space (the result expresses the Euler characteristic in terms of the principal curvatures, namely the eigenvalues of the second fundamental form). The proof uses the Gauss map from the manifold to the unit sphere.

However, hypersurfaces of Euclidean space are very special. For example, $\mathbf{CP}^n$ cannot be embedded into $\mathbf{R}^{2n+1}$ when $n \geq 2$. Note that it is reasonable to exclude odd-dimensional closed manifolds in a generalization of the Gauss–Bonnet theorem, since they have Euler characteristic zero.

In 1940, Allendoerfer [All40] and Fenchel [Fen40] used this result together with Weyl's tube formula to prove a version of Gauss–Bonnet for even-dimensional closed submanifolds of Euclidean space, where the integrand is a curvature invariant of the manifold. By the Nash embedding theorem, the assumption that the manifold be embedded in Euclidean space can be removed.

This proof is anachronistic, since the generalized Gauss–Bonnet theorem was proved before the Nash embedding theorem, by Allendoerfer and Weil [AW43] in 1943. Their proof used a weak version of the Nash embedding theorem which states that every Riemannian manifold has a local isometric embedding into Euclidean space. Allendoerfer and Weil embed the even-dimensional manifold one piece at a time into Euclidean space, and they give a complicated argument to patch the pieces together.

Finally, in 1945, Chern [Che45] gave a proof of the Gauss–Bonnet theorem that is independent of a choice of embedding. The theorem is now known as Chern–Gauss–Bonnet.

References

[All40] Carl B Allendoerfer. "The Euler number of a Riemann manifold". *Amer. J. Math.* 62.1/4 (1940), p. 243.

[Asp85] William Aspray. *The mathematics community at Princeton before 1930.* Manuscript, transcript number 29 (PMC29). Princeton University, 1985.

[AW43] Carl B Allendoerfer and André Weil. "The Gauss-Bonnet theorem for Riemannian polyhedra". en. *Trans. Am. Math. Soc.* 53.1 (1943), pp. 101–129.

[Che45] Shiing-shen Chern. "On the Curvatura Integra in a Riemannian Manifold". *Annals of Mathematics* 46.4 (1945), pp. 674–684.

[Fen40] W Fenchel. "On total curvatures of Riemannian manifolds: I". *J. Lond. Math. Soc. (2)* s1-15.1 (Jan. 1940), pp. 15–22.

[Gra03] Alfred Gray. *Tubes*. en. Progress in Mathematics. Basel, Switzerland: Springer, Nov. 2003.

[Hop26] Heinz Hopf. "Über die Curvatura integra geschlossener Hyperflächen". de. *Math. Ann.* 95.1 (Dec. 1926), pp. 340–367.

[Hot29] Harold Hotelling. "Stability in Competition". *Econ. J. (London)* 39.153 (Mar. 1929), p. 41.

[Hot39] Harold Hotelling. "Tubes and spheres in n-spaces, and a class of statistical problems". *Amer. J. Math.* 61.2 (Apr. 1939), p. 440.

[Ste13] Jakob Steiner. "Ueber parallele Flächen". In: *Jacob Steiner's Gesammelte Werke: Herausgegeben auf Veranlassung der königlich preussischen Akademie der Wissenschaften*. Ed. by Karl Weierstrass. Cambridge Library Collection - Mathematics. Cambridge University Press, 2013, pp. 171–176.

[Wey39] Hermann Weyl. "On the Volume of Tubes". *Amer. J. Math.* 61.2 (Apr. 1939), p. 461.